Lecture 2 Functions II

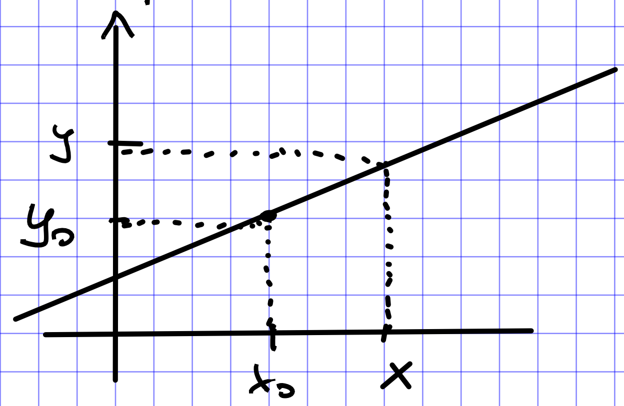
Recall from Lecture 1

Polynomials

1) Linear functions → Slope a & point (x_0, y_0)

$$y = a(x - x_0) + y_0$$

$$a = \frac{y - y_0}{x - x_0}$$



2) Quadratic functions

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Today we will see

1) Examples of other function

- Rational functions
- Power functions
- Exponential function
- Logarithm

2) Measure the slope of a function at a point.

Polynomials II $\quad [a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0]$, $\underbrace{a_0, a_1, \dots, a_n}$ real coefficients

$n \equiv$ degree

Factoring polynomials

To calculate $\frac{5}{4}$ we can perform the division algorithm to say that

$$5 = \underbrace{1}_{\text{quotient}} \cdot 4 + \underbrace{1}_{\text{remainder}}$$

Whenever we have two polynomials $P(x)$, $Q(x)$ with degrees m, and n & $m > n$ we can find two polynomials $q(x) \equiv$ quotient

$r(x) \equiv$ remainder (degree $r < n$)

such that

$$P(x) = q(x)\,Q(x) + r(x)$$

So

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

$$\frac{x^2 - 1}{x - 1}$$

$$x^2 - 1^2 = (x+1)(x-1)$$

A function of the type $\frac{P(x)}{Q(x)}$ is called a rational function

$$\frac{x^2 - 2}{x - 1} = x + 1 + \frac{(-1)}{x - 1}$$

- 2 is a factor of 6 if ($6 = 3\cdot 2 + \underline{\underline{0}}$) the remainder of $\frac{6}{2}$ is zero.
(6 is divisible by 2)
- 7 is not divisible by 2 because $7 = 3\cdot 2 + \underline{\underline{1}}$ (the remainder is not zero)

We say that $Q(x)$ is a factor of $P(x)$ if

$$P(x) = q(x)\cdot Q(x).$$

Example

i) $p(x) = x^2 - 1 = (x-1)(x+1)$ so $x-1$ is a factor of x^2-1

$(p(1) = 1^2 - 1 = 0 ; \quad p(-1) = (-1)^2 - 1 = 1 - 1 = 0)$

Fact: $(x-a)$ is a factor of $P(x)$ if and only if $P(a) = 0$

i.e. a is a zero (root) of $P(x)$

ii) $2x^5 + x^4 + x^3 - 4 = q(x)$

Note that
$$\left.\begin{aligned} q(1) &= 2\cdot 1^5 + 1^4 + 1^3 - 4 \\ &= 0 \end{aligned}\right\}$$
$x-1$ is a factor of $q(x)$

<u>Theorem</u> Every polynomial can be written as a product of polynomials of degree one, and polynomials of degree two <u>without real roots</u> (irreducible)

Factoring a polynomial of degree 2

We know that

$p(x) = ax^2+bx+c$ $\quad ax^2+bx+c=0 \iff x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

1) Note that if b^2-4ac ($=\Delta$ for Discriminant) is negative, $\sqrt{b^2-4ac}$ is not a real value

I.e. $p(x)$ is irreducible.

Example: $x^2+1=0 \iff x=\pm\sqrt{-1}$, but $\sqrt{-1}$ is not real!!

So x^2+1 has no real roots (irreducible)

2). If $b^2-4ac=0$ (It has a double root)

$$x^2+2x+1=(x+1)(x+1)=(x+1)^2$$

$$b^2-4ac=2^2-4\cdot1\cdot1=0$$

3). If $b^2-4ac>0$, it has two different roots. x_+ and x_-

$$x^2-1=0 \iff x=\pm1$$

$$\left[ax^2+bx+c=0 \iff x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad // \; a=1,\ b=0,\ c=-1\right]$$

$$x^2-1=(x+1)(x-1)$$

In the cases 2) & 3) if we denote the roots x_+ & x_- (in the case 2) $x_+ = x_-$) we can then write

$$\left[ax^2+bx+c = a(x-x_+)(x-x_-) \right]$$

__Examples__

i) $2x^2-2 = 2(x^2-1) = 2(x-1)(x+1)$

ii) $2x^2-8x+6 = 2\left(\overbrace{x^2-4x+3}\right) = 2(x-1)(x-3)$

We want to factorise x^2-4x+3 (i.e. we want to write

$p(x) = x^2-4x+3 = (x-a)(x-b)$)

So we need to find a, b such that $\underbrace{p(a)=0}$ & $\underbrace{p(b)=0}$

i.e. we need to solve

$$x^2-4x+3=0 \iff x = \frac{-(-4) \pm \sqrt{(-4)^2-4\cdot 1\cdot 3}}{2\cdot 1}$$

$$= \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm \sqrt{4}}{2}$$

$$= \frac{4 \pm 2}{2} = 2 \pm 1 = \begin{cases} 3 \\ 1 \end{cases}$$

$\Big\{$ So $a=1$ & $b=3$ are the two roots of $p(x)$

We can then write

$$\underbrace{x^2-4x+3} = (x-1)(x-3)$$

Trick

If P is a polynomial

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n$$

such that all its coefficients are integers

If k is an integer $(k = -3, -2, -1, 1, 2, 3, \ldots)$ zero of P

(i.e. $P(k) = 0$) then k is a divisor of a_0

Example

i) $p(x) = x^3 - 1$ is a polynomial with integer coefficients

If it has an integer zero k, $(P(k)=0)$ k has to be a divisor of -1. So k can be either 1 or -1

Note that

a) $P(-1) = (-1)^3 - 1 = -2 \neq 0$ so -1 is not a root of $P(x)$

b) $P(1) = 1^3 - 1 = 0$, so 1 is a root of $P(x)$.

So $x-1$ is a factor for $P(x)$ so we can try to divide $\frac{P(x)}{x-1}$

ii) $P(x) = x^5 + 2x - 3$

Divisors of $-3 = \pm 1, \pm 3$

$P(1) = 1^5 + 2 \cdot 1 - 3 = 0$

$P(-1) = (-1)^5 - 2 - 3 = -6 \neq 0$

$P(x) = x^2 + 1$

Divisors of 1: 1, -1

$P(1) = 2$

$P(-1) = 2$

$\mathbb{N}$ • Natural numbers (naturliga tal)

$0, 1, 2, 3, 4, \ldots \ldots$

$\mathbb{Z}$ • Integer numbers (heltal)

$\ldots -4, -3, -2, -1, 0, 1, 2, 3, \ldots$

$2x^2 - 3x + 4$ polynomial with integer coefficients

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$\frac{1}{2}x + 1$ has no integer coefficients.

$\mathbb{Q}$ • Rational numbers

$(\mathbb{Z})$ $3/2, 1/2, -5/7, \ldots 1, 2, -3,$

$\mathbb{R}$ • Real numbers

$\mathbb{Q}, \pi, e$

We can then try to divide $P(x)$ by $x-1$ by a similar method as we do for numbers

$$\begin{array}{l|l} x^3+0\cdot x^2+0\cdot x-1 & x-1 \\ \hline x^3+(-x^2) & x^2+x+1 \end{array}$$

$0 \quad x^2+0\cdot x-1$

x^2-x

$x-1$

$x-1$

0

$$\begin{array}{l|l} 5 & 2 \\ \hline 4 & 2 \end{array}$$

$5-4=1$

$$\begin{array}{l|l} 51 & 2 \\ \hline 4 & 25 \end{array}$$

11

10

1

$$\frac{51}{2}=25+\frac{1}{2}$$

$$\frac{x^3-1}{x-1}=x^2+x+1.$$

Example II Calculate

$$\frac{(-x^3+4x^2-x-7)}{x-2}$$

$$\begin{array}{r|l} & x-2 \\ \hline -x^3+4x^2-x-7 & \\ \hline \end{array}$$

All real numbers larger than 0

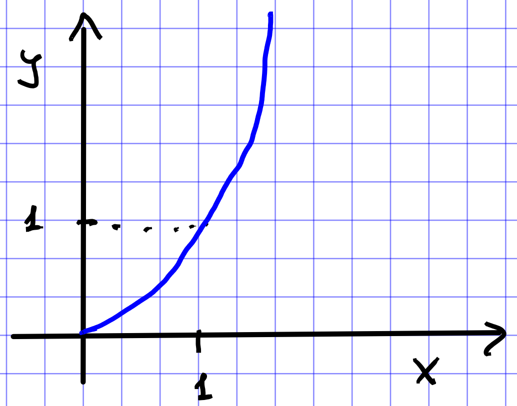
Power functions

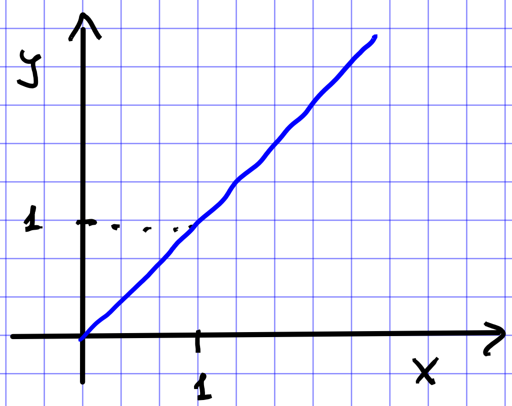
Given $A, r \in \mathbb{R}$ we define ($f : \overbrace{(0,+\infty)} \longrightarrow \mathbb{R}$) the power function

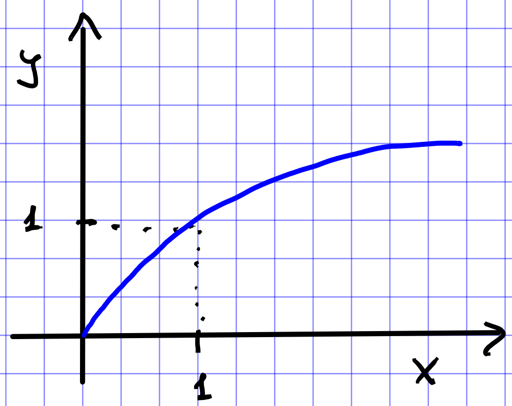
$$\boxed{f(x) = Ax^r}$$

Examples

i) $f(x) = x^2$ ii) $f(x) = x$ ii) $f(x) = x^{\frac{1}{2}} = \sqrt{x}$







$\underline{\underline{f(x) = x^r}}$

$r > 1$ $r = 1$ $0 < r < 1$

Notation Given $a > 0$ and $n = 1, 2, 3, 4, \dots$

$a^{1/n} = \sqrt[n]{a}$ is the unique positive number such that
(nth root of a)

$$(a^{1/n})^n = a$$

Example

i) $\sqrt[2]{4} =: \sqrt{4} = 4^{1/2}$ is the unique positive number b such that $b^2 = 4$. Therefore $b = 2$ ($\sqrt{4} = 2$) as $2^2 = 4$)

ii) $\sqrt[3]{8} = 8^{1/3}$ is the unique positive number b such that $b^3 = 8$. Therefore $b = 2$ ($2 \cdot 2 \cdot 2 = 2^3 = 8$)

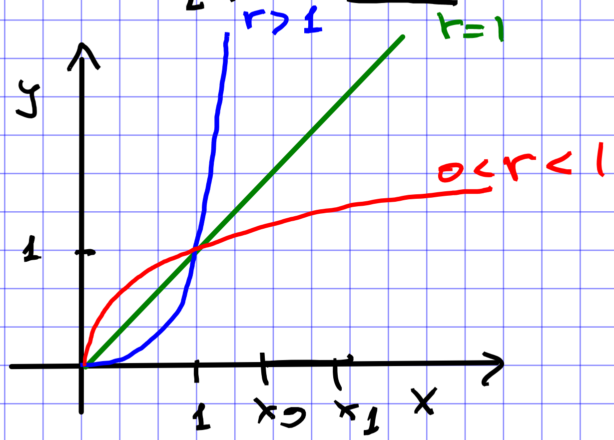
Properties

i) $2^2 \cdot 2^3 \left(= 4 \cdot 8 = \overset{2^5}{\overset{\shortparallel}{32}}\right) = 2^{2+3}$ i) $b^r \cdot b^s = b^{r+s}$

ii) $(2^3)^2 \left(= (8)^2 = \underset{2^6}{\underset{\shortparallel}{64}}\right) = 2^{3\cdot 2} = 2^6$ ii) $(b^r)^s = b^{r \cdot s}$

Note $\left(\frac{1}{b}\right)^r = b^{-r}$ Example $\left(\frac{1}{2}\right)^3 = 2^{-3} = (2^{-1})^3$

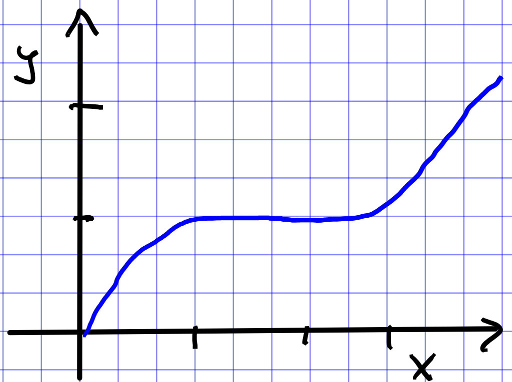
The function $f(x) = x^r$ with $r > 0$ is strictly increasing

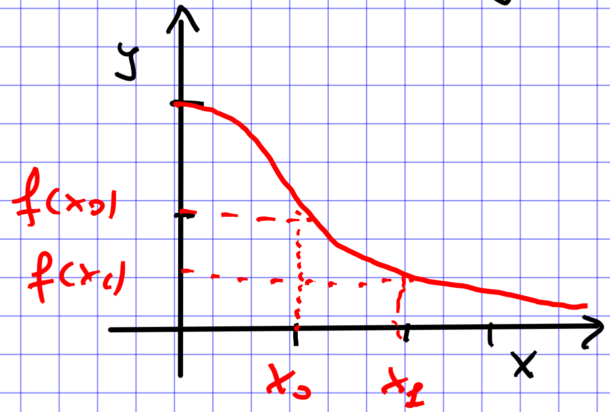


Def A function f is (strictly) increasing if for $x_0 < x_1$ we have that

$$f(x_0) \leq f(x_1) \; (f(x_0) < f(x_1))$$

Note: The function with graph as below is increasing but not strictly increasing





Def A function f is (strictly) decreasing if for $x_0 < x_1$ we have that

$$f(x_0) \geq f(x_1) \; (f(x_0) > f(x_1))$$

Exponential function

The exponential function with base $b > 0$ (fixed) is the one given by

$$f : \mathbb{R} \longrightarrow (0, +\infty)$$
$$x \longmapsto f(x) = b^x \qquad // \text{ Ex. } x \mapsto 2^x$$

In practice, you will encounter functions of the type $A b^x$ with $b > 0$ and $A > 0$ given.

Example

(**Compound Interest**) A savings account of $\$K$ that increases by $p\%$ interest each year will have increased after t years to $K(1 + p/100)^t$ (see Section 1.2). According to this formula with $K = 1$, a deposit of $\$1$ earning interest at 8% per year ($p = 8$) will have increased after t years to $(1 + 8/100)^t = 1.08^t$.

$k b^t$

$b = \left(1 + \frac{p}{100}\right) > 1$

Table 1 How \$1 of savings increases with time at 8% annual interest

t	1	2	5	10	20	30	50	100	200
$(1.08)^t$	1.08	1.17	1.47	2.16	4.66	10.06	46.90	2199.76	4 838 949.60

After 30 years, \$1 of savings has increased to more than \$10, and after 200 years, it has grown to more than \$4.8 million! Observe that the expression 1.08^t defines an exponential function of the type (1) with $a = 1.08$. Even if a is only slightly larger than 1, $f(t)$ will increase very quickly when t is large.

(Continuous Depreciation) Each year the value of most assets such as cars, stereo equipment, or furniture decreases, or *depreciates*. If the value of an asset is assumed to decrease by a fixed percentage each year, then the depreciation is called *continuous*. (Straight line depreciation was discussed in Problem 4.5.5.)

Assume that a car, which at time $t = 0$ has the value P_0, depreciates at the rate of 20% each year over a 5 year period. What is its value $A(t)$ at time t, for $t = 1, 2, 3, 4, 5$?

$P_0 b^t$

$b = (1 - \frac{20}{100})$

$b < 1$

Solution: After 1 year its value is $P_0 - (20P_0/100) = P_0(1 - 20/100) = P_0(0.8)^1$. Thereafter it depreciates each subsequent year by the factor 0.8. Thus, after t years, its value is $A(t) = P_0(0.8)^t$. In particular, $A(5) = P_0(0.8)^5 \approx 0.32P_0$, so after 5 years its value has decreased to about 32% of its original value.

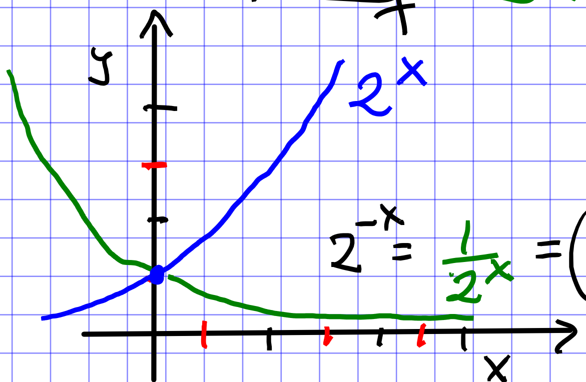
Properties

i) $b^x \cdot b^y = b^{x+y}$

ii) $(b^x)^y = b^{x \cdot y}$

iii) If $b > 1$ the function $x \longmapsto b^x$ is strictly increasing

iv) If $b < 1$ the function $x \longmapsto b^x$ is strictly decreasing



$2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$

Def. If $b = e \simeq 2.71828\ldots$ the function $f(x) = e^x$ is called the natural exponential function.

Doubling time

If $b>1$ the doubling time is the time the exponential function $f(t) = Ab^t$ to double its value.

(note: Here I'm using t for independent variable instead of the usual x)

That is. If at time t_0 the value is $f(t_0) = Ab^{t_0}$, t_* is the time such that

$$f(t_0+t_*) = 2f(t_0)$$

Note that

$$A\overbrace{b^{t_0+t_*}}^{b^{t_0}\cdot b^{t_*}} = 2Ab^{t_0} \iff$$

$$\left[b^{t_0+t_*} = b^{t_0}\, b^{t_*}\right] \iff \left[b^{t_*} = 2\right]$$

[Dividing by A and b^{t_0} in both sides]

Question: How can we solve this equation i.e. find t_* satisfying that $b^{t_*} = 2$?

Example

i) If $b=4$. We want to find t_* such that

$$4^{t_*}=2$$

$$\begin{cases} 4^{t_*}=2 \iff [4=2^2]\ (2^2)^{t_*}=2 \iff \underbrace{2}^{2t_*}=2=\underbrace{2}^{1} \\ \iff 2t_*=1 \iff \boxed{t_*=\frac{1}{2}},\quad [4^{\frac{1}{2}}=\sqrt{4}=2] \end{cases}$$

ii) If $b=\frac{1}{2}$, we want to find t_* such that

$$\left(\frac{1}{2}\right)^{t_*}=2.$$

$$\left(\frac{1}{2}\right)^{t_*}=2 \iff (2^{-1})^{t_*}=2 \iff 2^{-t_*}=2^{1} \iff \boxed{-t_*=1}$$

$\frac{1}{2}=2^{-1}$ $\quad$ $[(2^r)^s=2^{r\cdot s}]$ $\quad$ $\iff \boxed{t_*=-1}$

$\left(\frac{1}{b}=b^{-1}\right)$

2 $\quad$ 2^x

[Divide both sides by -1]

$-t_*=3 \Rightarrow$ [Divide by -1 both sides] $\quad t_*=\frac{3}{-1}=-3$

$$\left(\begin{matrix} ax+b=0 \\ a\neq 0 \end{matrix} \Rightarrow ax=-b \Rightarrow [x=-b/a] \right)$$

$\left[b^{t_*} = 2 \right] \left[t_* = \log_b 2 \right]$

Logarithmic function.

Fixed $b>0$, and given $x>0$

we say that t is the logarithm to base b of x

if it satisfies that

$$b^t = x$$

We denote it by $\log_b x$.

$$\boxed{b^{\log_b x} = x}$$

Examples Calculate the following logarithms

i) $\log_2 16$ ii) $\log_{10}\left(\frac{1}{100}\right)$ iii) $\log_3\left(\sqrt[5]{9}\right)$

i) $\log_2 16$ is the solution to the equation

$$2^t = 16 \quad / \quad 2^3 = 2\cdot 2\cdot 2 = 8,\ 2^4 = 16$$

$$\log_2 16 = 4$$

ii) $\log_{10}\left(\frac{1}{100}\right)$ is the solution t to $10^t = \frac{1}{100} = \frac{1}{10^2}$

$$\log_{10}\left(\frac{1}{100}\right) = -2. \qquad = 10^{-2}$$

$\log_3(\sqrt[5]{9})$ is the solution t

to $3^t = \sqrt[5]{9}$

Note

$$\sqrt[5]{9} = (9)^{1/5} = (3^2)^{1/5} = 3^{\frac{2}{5}}$$

So we look for t such that

$$3^t = 3^{2/5}$$

Hence $t = 2/5$

If $b = e$, we simply write

$$\boxed{\ln x := \log_e x}$$

and it's called the natural logarithm of x

That is

$$e^{\ln x} = x \quad // \quad b^{\log_b x} = x$$

Properties

i) $\log_b 1 = 0 \quad (b^t = 1)$

ii) $\log_b b = 1 \quad b^t = b$

iv) $\log_b (b^x) = x$

v) $\log_b (x \cdot y) = \log_b x + \log_b y$

vi) $\log_b (x^p) = p \log_b x$

Important Remark:

$$b^{\log_b x} = x \Rightarrow \ln x = \ln (b^{\log_b x}) =$$