Course MM1005

Lecture 3: Other functions

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Questions?

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Lecture Goal and Outcome

Goal: Review of useful functions often used in Economics (this is a deepening of concepts you should have seen in the Matte 2b/c and Matte 3 classes)

Learning Outcome: After today you will be able to

- Find zeros of polynomial functions
- Solve problmes like the following:

Problem

An investment gives 2% interest every year. After how many years has an initial investment doubled? Tripled?

Why you should care

Many man made and naturally occurring phenomena, including city sizes, incomes, word frequencies, and earthquake magnitudes, are distributed according to a power-law distribution.

Example

The probability that a person has an income greater than x is $P(x) = Cx^a$ with C a positive real number and a a negative real number.

You have a set of data and you want to find C and a that fits the data best.

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You have a set of data and you want to find C and a that fits the data best. Apply the logarithm on the data set and you will see that your data now fit in a line (and you can use linear regression).

$$\log(P(x)) = a\log(x) + \log(C),$$

Lecture Plan

- Polynomials
- Power functions
- Exponential functions
- Logarithmic functions

Section 1
Polynomials

Polynomial functions

A polynomial function is a function of the form

$$p(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0,$$

where d is a natural number and the coefficient c_i's are real numbers.

Polynomial functions

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$$p(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0,$$

where d is a natural number and the coefficient c_i's are real numbers.

- If d is the highest power appearing in the law of p we say that p is a polynomial of degree d (we write $\deg(p) = d$).
- The monomial of highest degree $c_d x^d$ is called the leading term of p.
- The coefficient c_d is called the leading coefficient of p.
- The coefficient c_0 is called the constant coefficient of p.

Convention

The identically 0 function ($0(x) = 0$) is a polynomial and we set $\deg(0) = -\infty$.

Examples

- Linear functions are polynomial functions. What is their degree?
- Quadratic functions are polynomial functions. What is their degree?
- Constant functions are polynomial functions. What is their degree?
- $x^3 - 1$
- $x^{1000} + 2x^{100} + 3x$

Polynomial multiplication

If one multiplies two polynomial functions p and q, one gets again a polynomial function.

Example

Multiply $(x+3)^2$ and $(x+1)$.

Useful fact:

$$\deg(pq) = \deg(p) + \deg(q),$$

Roots

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- Polynomials of even degree might or not have real roots.

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How do we find the roots?

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Finding roots of polynomials and in general zeroes of functions will be really important when we will speak about optimization and study of functions (Lecture 7 and usually an exam problem)

Polynomial Division

Given a polynomial p, and a nonzero polynomial q it is always possible to find two polynomials k and r, with r either 0 or $\deg(r) < \deg(q)$ such that

$$p(x) = k(x)q(x) + r(x),$$

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Application to finding roots

The real number a is a root of p if, and only if,

$$p(x) = k(x)(x - a) + 0,$$

Example

Find the roots of $p(x) = x^3 - 11x + 6$.

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- Observe that 3 is a root.
- Divide $p(x)$ by $(x - 3)$.
- We get a polynomial of degree 2 for which we can always find the roots (if there are)

The Hunt for rational roots

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Useful fact

Let $p(x)$ be a polynomials with integer coefficients (that is $c_i \in \mathbb{Z}$). If rational number $\frac{m}{n}$ is a root for p, then we have that n divides the leading coefficient and m divides the constant term.

Example

Find all the rational roots of $p(x) = x^4 - 2x^3 - 14x^2 + 30x + 9$.

Summary

To find the roots of a polynomial of high degree:

- Look for easy roots.
- Divide.
- Repeat till you get a polynomial of degree 1 or 2.

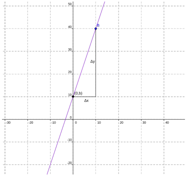
Questions?

Section 2
Power Functions

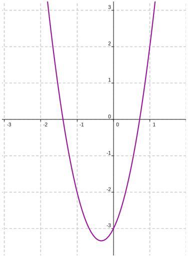
Increasing/Decreasing functions

- A function $f(x)$ is said to be increasing on an interval I if for every x and $y \in I$ we have that $x < y$ implies that $f(x) \leq f(y)$
- A function $f(x)$ is said to be strictly increasing on an interval I if for every x and $y \in I$ we have that $x < y$ implies that $f(x) < f(y)$
- A function $f(x)$ is said to be decreasing on an interval I if for every x and $y \in I$ we have that $x < y$ implies that $f(x) \geq f(y)$
- A function $f(x)$ is said to be strictly decreasing on an interval I if for every x and $y \in I$ we have that $x < y$ implies that $f(x) > f(y)$

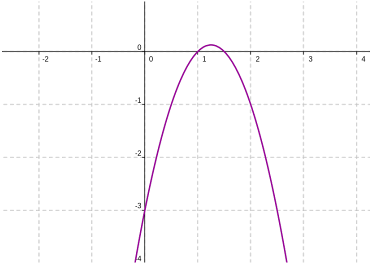
Examples



$3x + 10$



$a > 0$



$a < 0$

Questions?

Power functions

A power function is a function of the form

$$f(x) = Cx^q$$

Where C is a real number and q is a rational number.

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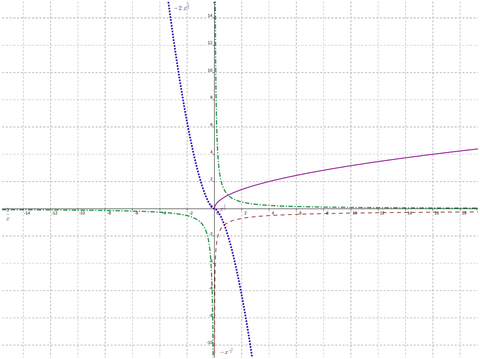

- The domain can be either $\mathbb{R}$, $\mathbb{R}^*$, $\mathbb{R}_{\geq 0}$ or $\mathbb{R}^+$, depending from q.
- If $x \geq 0$ they are either always increasing or decreasing depending from C.

Questions?

Section 3
Exponential functions

Exponential function

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Where C is a real number and a is a positive real number.

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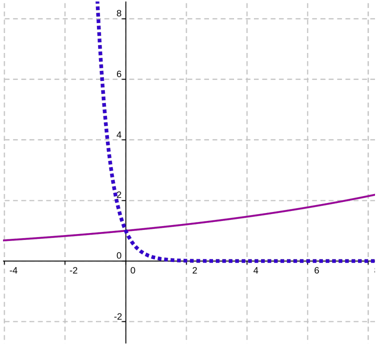

- The domain is all $\mathbb{R}$
- Different behavior if $a > 1$ or $a < 1$.
- They are always increasing or decreasing.
- They have the same positivity of C

The number e

Suppose you invest a capital K for 1 year at interest rate x. After one year you have

$$K\left(1+\frac{x}{1}\right)^{1},$$

If you disinvest after 6 month and reinvest immediately you will have

$$K\left(1+\frac{x}{2}\right)^{2},$$

With n withdrawal and deposit:

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We will see that this number is really useful when integrating and deriving. Thus e^x is the best exponential function.

From a to e

The range of e^x is the whole positive real line. So if a is a positive real number we can write $a = e^r$ for some r. Thus

$$f(x) = Ca^x = C(e^r)^x = Ce^{rx},$$

Exercise

Find a and C such that $f(x) = Ca^x$ fits the data points $(-1, 2/3)$ and $(3, 54)$

Questions?

Section 4
Logarithm

Logarithm

Since a^x is always strictly decreasing, and its range is the whole positive real line, for every positive real number b the equation

$$a^x = b$$

has a unique solution, denoted by $\log_a(b)$ (logarithm in base a of b). For every positive real number a we get a function

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- When the base is 10 we write simply $\log(x)$.
- When the base is e we write $\ln(x)$, this is the natural logarithm of x.

Properties of the logarithm

- For every real number x we have that $\log_a(a^x) = x$.
- For every positive real number x we have that $a^{\log_a(x)} = x$.
- For every a positive real number

$$\log_a(1) = 0,$$

- $\log_a(xy) = \log_a(x) + \log_b(x)$
- $\log_a(x/y) = \log_a(x) - \log_a(y)$
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Attention!!!

There is no "$\log_a$" in your calculator, but only $\ln$ or $\log$...

$$\boxed{\log_a(x) = \frac{\log_b(x)}{\log_b(a)} = \frac{\ln(x)}{\ln(a)}}$$

Examples

- Compute $\log_{2,01}(2)$ with your calculator (using the change of base formula).
- Simplify

$$\frac{\ln(2^{3/8})}{\ln(2^{-1/2})}$$

- Let $f(t) = 200(0{,}8)^t$. For which t we have $f(t) = 100$?

Back to our problem

Problem

An investment gives 2% interest every year. After how many years has an initial investment doubled? Tripled?

Questions?

Thank you for your attention!