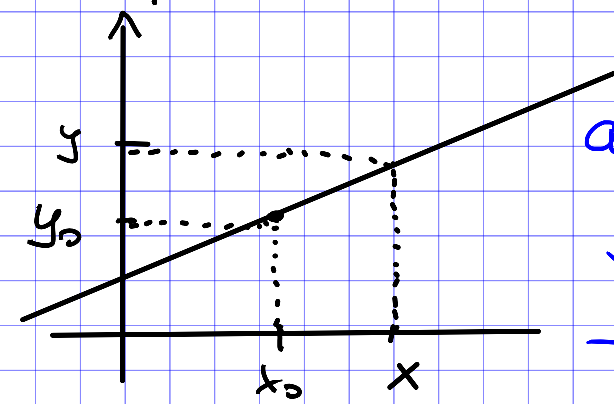
Lecture 3

The story so far

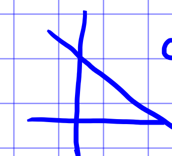
Linear functions → Slope a & point (x_0, y_0)

$$\boxed{y = a(x - x_0) + y_0}$$

$$a = \frac{y - y_0}{x - x_0}$$



$a > 0$



$a < 0$

Examples of functions

- Polynomials
- Rational functions
- Power functions
- Exponential functions
- Logarithms

Lecture 2

(1) $\boxed{b^{\log_b x} = x}$ $\xrightarrow{\text{Taking } \log_a \text{ in both sides}}$

$$\log_a x = \log_a\left(b^{\log_b x}\right)$$

$$= (\log_b x)(\log_a b)$$

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$$

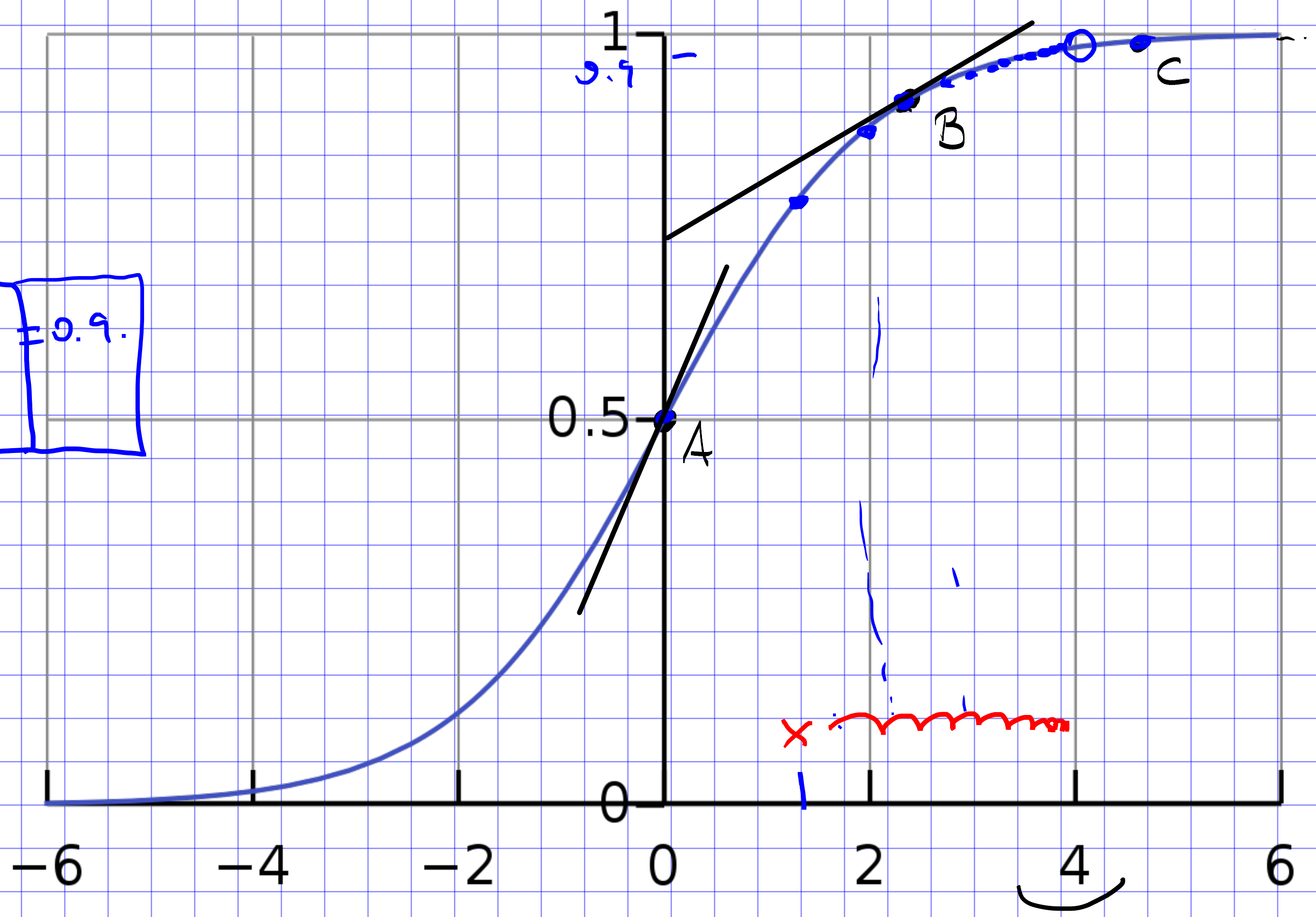
(2) $\log_a(y^p) = p \log_a y$

(3) $\boxed{\ln z := \log_e z}$

Today

→ $\boxed{\text{Derivatives \& slope}}$

→



$\lim\limits_{x\to 4} f(x) = 0.9$

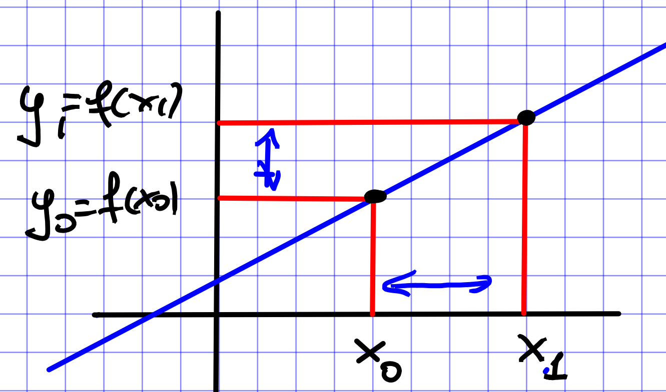
On which of the three points is the graph steepest?

A)
B)
C)

How can we measure how steep is a function at a given point?

Slopes: How can we measure how steep is a function at a given point?

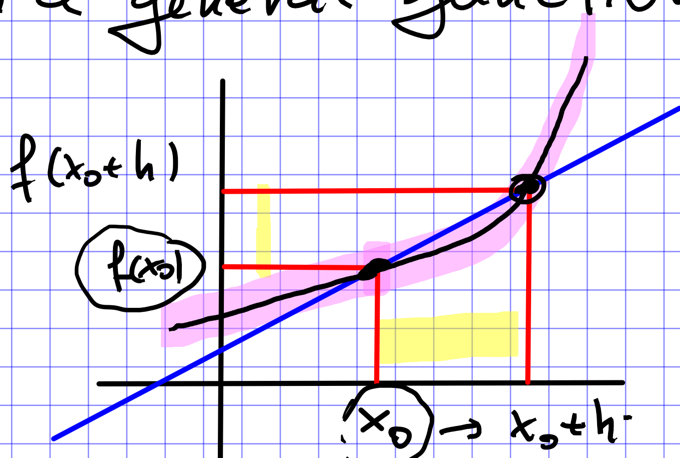
If the function is a line the steepness is codified on its slope



$$y = a(x - x_0) + y_0$$

$$\left[a = \frac{y_1 - y_0}{x_1 - x_0} \right]$$

For a general function f, what is the steepness of its graph at a point x_0?



$$\frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0}$$

Idea: Measure the steepness of lines through $(x_0, f(x_0))$, $(x_0+h, f(x_0+h))$ when h is "infinitely small"

That is, for given h close to 0 (e.g. $h=10^{-k}$, $k=1,2,3,\dots$)
we measure

$$\left[\frac{f(x_0+h)-f(x_0)}{h}\right]$$

and let h get *infinitely* closer to 0.

The value it approximates is denoted by $f'(x_0)$
and it's called the derivative of f at x_0

See the following link for an illustration
for the function $[f(x)=x^2]$

In this case $f(x)=x^2$ we have

If h is "infinitely small"

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{(x_0+h)^2-x_0^2}{h} = \frac{\cancel{x_0^2}+2x_0h+h^2-\cancel{x_0^2}}{h}$$

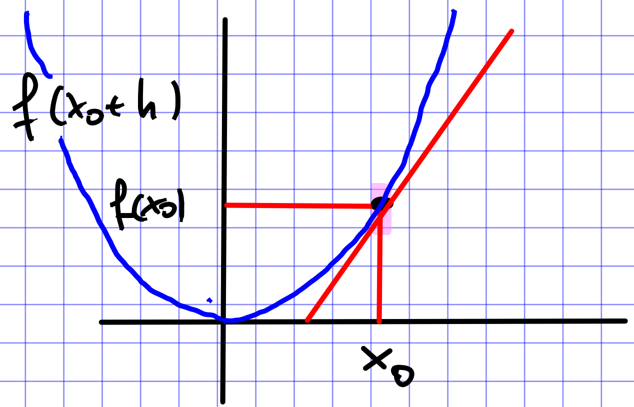
$$= \frac{2x_0h+h^2}{h} = 2x_0+h$$

$$\left[h = 1/10000000\right]$$

$\frac{f(x_0+h)-f(x_0)}{h}$ gets closer and closer to $2x_0$ as
h gets closer and closer to 0

$$\boxed{\begin{aligned} f(x)&=x^2 \\ f'(x)&=2x \end{aligned}}$$

$[f'(1)=2]$



The line with equation

$$y = f'(x_0)(x - x_0) + f(x_0)$$

i) goes through $(x_0, f(x_0))$
ii) has slope $f'(x_0)$

is called the tangent line to the graph of f at the point $(x_0, f(x_0))$

Notation

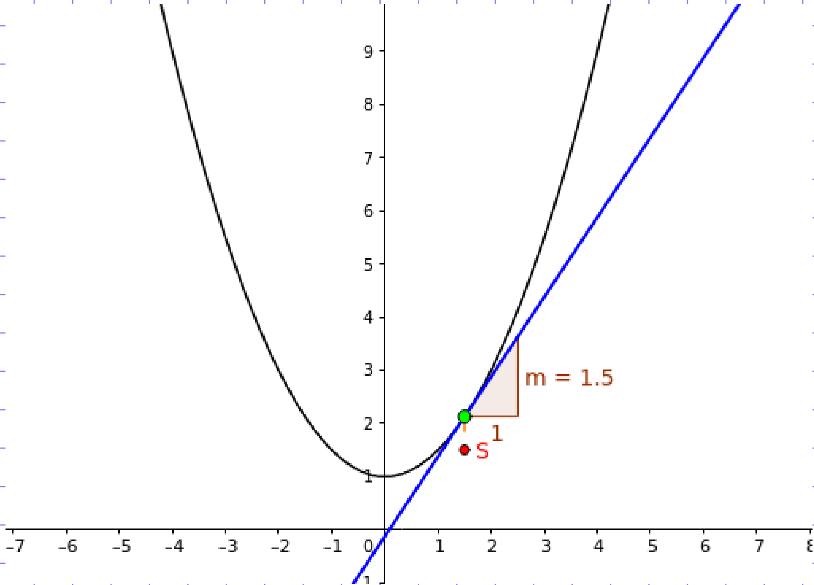
i) $f'(x_0) = \frac{df}{dx}(x_0)$ (differential notation)

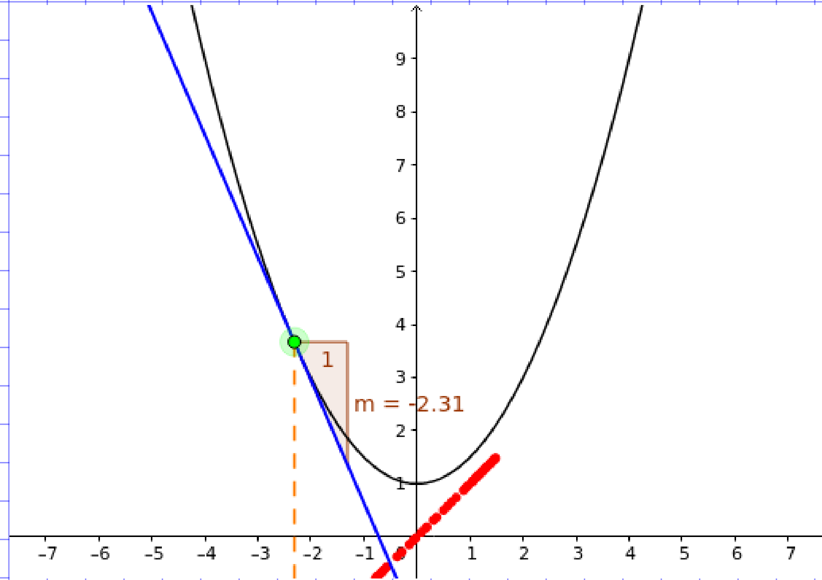
$= \dot{f}(x_0)$

Formally

$$f'(x_0) = \lim_{h \to 0} \overbrace{\frac{f(x_0+h) - f(x_0)}{h}}^{H(h)}$$

The quantity $H(h)$ gets closer and closer to $f'(x_0)$ as h gets closer and closer to 0





$f(x) = \frac{1}{2}x^2 + 1$

Theorem

If $f'(x) \geq 0$ on a whole interval $I \Longleftrightarrow f$ is increasing on I

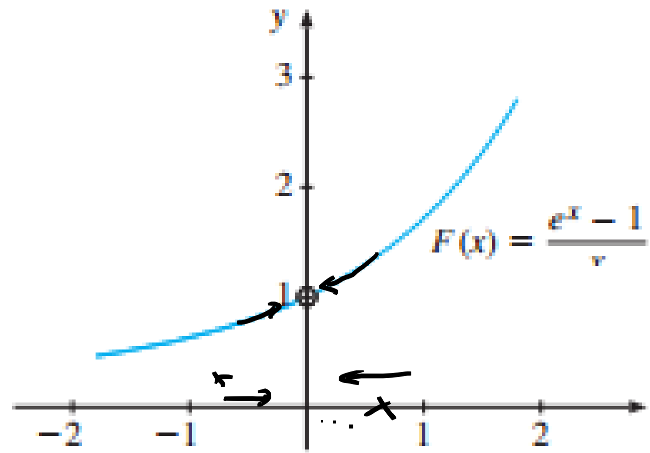
If $f'(x) \leq 0$ on a whole interval $I \Longleftrightarrow f$ is decreasing on I

Limits

Let $F(x) = \dfrac{e^x - 1}{x}$ for $x \neq 0$

x	-1	-0.1	-0.001	-0.0001	0.0	0.0001	0.001	0.1	1
$F(x)$	0.632	0.956	0.999	1.000	*	1.000	1.001	1.052	1.718

* not defined



We note that $F(x)$ approaches 1 as x gets closer and closer to 0

What happens with $F(x)$ as x gets closer to zero

So we write

$$1 = \lim_{x \to 0} \frac{e^x - 1}{x}$$

Definition We say that L is the limit of f as x tends to a if the values of $f(x)$ gets closer to L as x gets closer to a

We then write $L = \lim_{x \to a} f(x)$

Examples

i) $\lim_{h \to 0} \frac{(x_0+h)^2 - x_0^2}{h} = 2x_0$

ii) $\lim_{x \to a} x = a$ $\quad \lim_{x \to a} 2 = 2$ $\quad \left(\lim_{x \to a} A = A \right)$

A constant

iii) $\lim_{x \to a} x^n = a^n$

Properties of limits

i) $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

$$\lim_{x \to a} (x^2+2) = \lim_{x \to a} x^2 + \lim_{x \to 2} 2 = a^2+2$$

ii) $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

$$\lim_{x \to a} (2x^2+3) = \lim_{x \to a} 2x^2 + \lim_{x \to a} 3$$

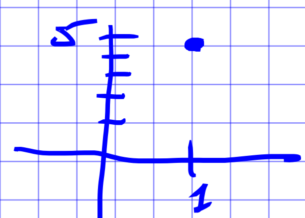
$$= \underbrace{\left(\lim_{x \to a} 2\right)}_{2} \left(\lim_{x \to a} x^2\right) + \lim_{x \to a} 3 = 2a^2+3$$

$$\lim_{x\to 1}\left(3x^5+2x^2-x+2\right) = 3\cdot 1^5+2\cdot 1^2-1+2$$
$$= 3+2-1+2 = 6$$

iii) $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}.$$ Provided $\boxed{\lim\limits_{x\to a} g(x)\neq 0}$

$$\lim_{x\to 1}\frac{5x^2+3x+2}{x+1} = \left[\text{Note that } \lim_{x\to 1}(x+1) = 1+1 = 2\neq 0\right]$$

$$= \frac{\lim\limits_{x\to 1}\left(5x^2+3x+2\right)}{\lim\limits_{x\to 1}(x+1)} = \frac{5\cdot 1^2+3\cdot 1+2}{1+1} = \frac{10}{2} = 5$$



(provided $\left(\lim\limits_{x\to a} f(x)\right)^r$ is defined)

iv) $$\lim_{x\to a}\left(f(x)\right)^r = \left(\lim_{x\to a} f(x)\right)^r$$

$$\lim_{x\to 0}\left(\sqrt{1+x}-1\right) = \lim_{x\to 0}(1+x)^{1/2} + \overbrace{\lim_{x\to 0}(-1)}^{-1} = \left(\lim_{x\to 0}(1+x)\right)^{1/2} - 1$$
$$= 1^{1/2}-1 = 1-1 = 0$$

What happens when $\left[\lim_{x\to a} g(x) = 0\right]$?

Assume first that $\left[\lim_{x\to a} f(x) = 0\right]$ too.

Strategy

Manipulate the expression of $\frac{f(x)}{g(x)}$ to simplify it

Examples

$$\lim_{x\to 2} \underbrace{\frac{3x^2+3x-18}{x-2}}_{\text{Rational function}}$$

Note that

$$\left[\lim_{x\to 2} 3x^2+3x-18 = 0 \quad \& \quad \lim_{x\to 2} x-2=0\right]$$

So we can't use the property iii above

- Make the polynomial division

or

- Factor the polynomials & simplify.

$$3x^2+3x-18 = 3\left(x^2+x-6\right) = 3(x-2)(x+3)$$

$$x^2+x-6=0 \iff x = \frac{-1\pm\sqrt{1+24}}{2} = \frac{-1\pm5}{2} = \begin{cases} 2 \\ -3 \end{cases}$$

Now

$$\frac{3x^2+3x-18}{x-2} = \frac{3\cancel{(x-2)}(x+3)}{\cancel{x-2}}$$

$$= 3(x+3)$$

$$\lim_{x \to 2} \frac{3x^2+3x-18}{x-2} = \lim_{x \to 2} 3(x+3) = 3(2+3) = 15$$

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$$

Remember

i) $(a-b)(a+b) = a^2 - b^2$

ii) $(\sqrt{a})^2 = a \quad (a>0)$

$$\frac{\sqrt{1+x}-1}{x} = \frac{\sqrt{1+x}-1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

$$\underset{(i)}{=} \frac{(\sqrt{1+x})^2 - 1^2}{x(1+\sqrt{1+x})} \underset{(ii)}{=} \frac{(1+x)-1}{x(1+\sqrt{1+x})} = \frac{\cancel{x}}{\cancel{x}(1+\sqrt{1+x})} = \frac{1}{1+\sqrt{1+x}}$$

- $\lim_{x \to 0} (\sqrt{1+x}-1) = 0$
- $\lim_{x \to 0} x = 0$

- $\lim_{x \to 0} 1+(1+x)^{\frac{1}{2}} = 1+1 = 2$
- $\lim_{x \to 0} 1 = 1.$

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \to 0} \left(1+(1+x)^{\frac{1}{2}}\right)^{-1} = \left(\lim_{x \to 0} 1+(1+x)^{\frac{1}{2}}\right)^{-1} = 2^{-1} = \frac{1}{2}.$$

What happens when $\lim\limits_{x\to a} f(x) \neq 0$ but $\lim\limits_{x\to a} g(x) = 0$?

We say that $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \infty$ (∞ infinity)

("arbitrarily large)

Example (Idea of ∞)

- $\lim\limits_{x\to 0} \dfrac{1}{x} = \infty$

Assume that x is getting closer and closer to 0

$\dfrac{1}{\frac{1}{100}} = \dfrac{1}{10^{-2}} = (10^{-2})^{-1} = 10^{(-1)(-2)} = 10^2$

x	$1/100$	$1/1000$	$\cdots$	$1/10^{100}$	
$1/x$	100	1000		10^{100}	$\dashrightarrow +\infty$

x	$-1/100$	$-1/1000$	$\cdots$	$-1/10^{100}$	
$1/x$	-100	-1000		-10^{100}	$\dashrightarrow -\infty$

We have define the derivative of f as a limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Properties

i) $(f+g)'(x) = f'(x) + g'(x)$

ii) $[(f \cdot g)'(x) = f'(x)\, g(x) + f(x)\, g'(x)]$ (Leibnitz Rule) (Product rule)

$[$If g is a constant $g'(x) = 0$

$(2f)'(x) = 2 f'(x)]$

iii) If $g'(x) \neq 0$ & $g(x) \neq 0$

$$\left[\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\right]$$

One can prove that if $[f(x) = x^r$ then $f'(x) = r x^{r-1}]$

$(f(x) = x^2 \quad f'(x) = 2x^1)$

$$\boxed{f(x) = x^r \rightarrow f'(x) = r x^{r-1}}$$

Examples

i) $f(x) = 3x^4 + 2x^2 + 4x + 1$

$$\frac{d}{dx} f(x) \overset{(i)}{=} \frac{d}{dx}(3x^4) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(4x) + \overbrace{\frac{d}{dx}(1)}^{0}$$

$$= 3\frac{d}{dx}(x^4) + 2\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x) + 0$$

$$= 3\left[4x^3\right] + 2\left[2 \cdot \underbrace{x^1}_{x}\right] + 4\left(1 \cdot \underbrace{x^0}_{1}\right) = 12x^3 + 4x + 4$$

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ii) $f(x) = \frac{1}{x-2}$