Course MM1005

Lecture 4: Limits and Derivatives

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Questions?

Lecture Goal and Outcome

Stockholm University

Goal: Introduction to limits and derivatives (If you have taken it, you should have seen some of this material in your Matte 3 class)

Learning Outcome: After today you will be able to solve problems like the followings:

Problem

If the total saving of a country is a function $S(Y)$ of the national product Y, then $S'(Y)$ is called the marginal propensity to save, or MPS. Find the MPS for the following functions:

- $S(Y) = S_0 + sY$
- $S(Y) = 100 + 0{,}1Y + 0{,}0002Y^2$

Why you should care

Stockholm University

When you model economics quantity with functions, the derivatives of these function gives you useful information. They represents marginals and allow you to do marginal analysis.
Marginal analysis is an examination of the associated costs and potential benefits of specific business activities or financial decisions. The goal is to determine if the costs associated with the change in activity will result in a benefit that is sufficient enough to offset them. Instead of focusing on business output as a whole, the impact on the cost of producing an individual unit is most often observed as a point of comparison.

Lecture Plan

- Limits (6.5)
- Derivatives (6.1-6.3, 6.6-6.11)

Attention

We use a different order than in the book.

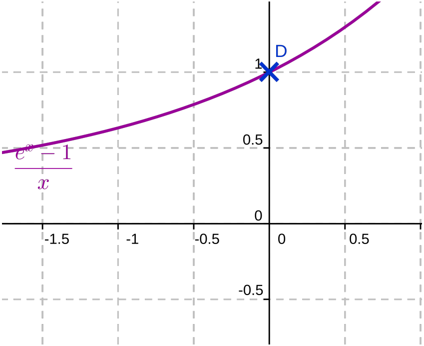
Section 1
Limits

Example

Consider the function

$$f(x) = \frac{e^x - 1}{x},$$

What is the function's domain? Let's us investigate what happens when the x get close to 0:



x	$f(x) \simeq$
-1	0.632
-0.1	0.956
-0.001	0.999
0	?
0.001	1.001
0.1	1.052
1	1.718

$$\lim_{x \to 0} f(x) = 1$$

Definition of limit

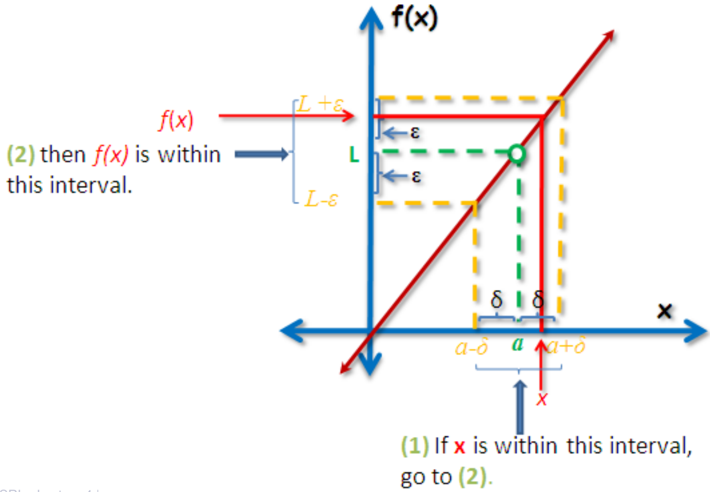
Definition (p. 277)

Let f be a function over an interval I, and let a be in I. If for every $\epsilon > 0$ we have that that there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in I$ is such that $|x - a| < \delta$, we say that f has limit L when x goes to a:

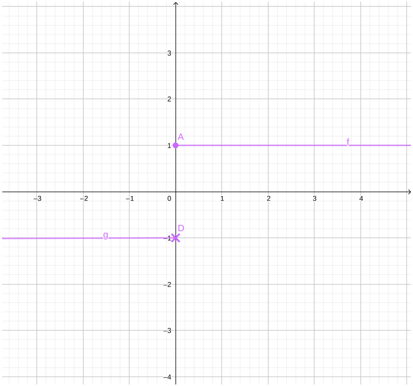
$$\lim_{x \to a} f(x) = L$$

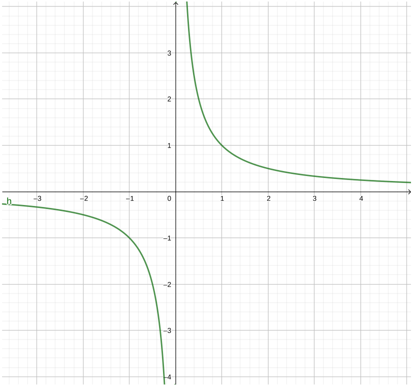
What does this mean?

Heuristically this means that when x get close to a $f(x)$ get close to L



Sometimes it does not exist





Rule for limits (to know!!!)

Let f and g two function on I and let

$$A := \lim_{x\to a} f(x), \quad B := \lim_{x\to a} g(x)$$

Then

- $\lim_{x\to a}(f(x) \pm g(x)) = A \pm B$
- $\lim_{x\to a}(f(x) \cdot g(x)) = A \cdot B$
- If c is a real number $\lim_{x\to A} cf(x) = cA$
- If $B \neq 0$, $\lim_{x\to a}(f(x)/g(x)) = A/B$
- If $f(x) < g(x)$ (or $f(x) \leq g(x)$) for every $x \in I$, then $A \leq B$

There is also a rule for the composition:
Suppose that $\lim_{x\to a} f(x) = A$ and that $\lim_{y\to A} g(y) = B$, then

$$\lim_{x\to a} g \circ f(x) = B$$

Exercises

- $\lim_{x\to 2} \frac{x^2-3x+2}{x-1}$
- $\lim_{x\to 1} \frac{x^2-3x+2}{x-1}$
- $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

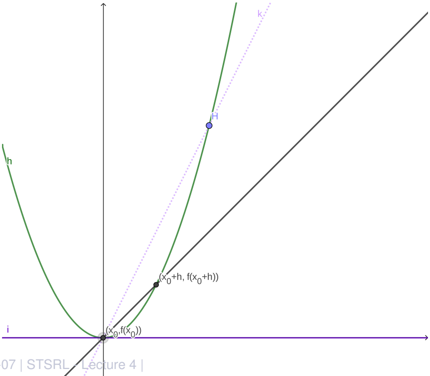
Questions?

Section 2
Derivatives

Tangents

Problem

We want to find the tangent line to a graph of a function $y = f(x)$ in a point $(x_0, f(x_0))$.



The slope of the secant is

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

Slope of the tangent

We have found that the slope of the secant line to the graph passing for $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h)$ is

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

If we let h go to 0 we should get the slope of the tangent line.

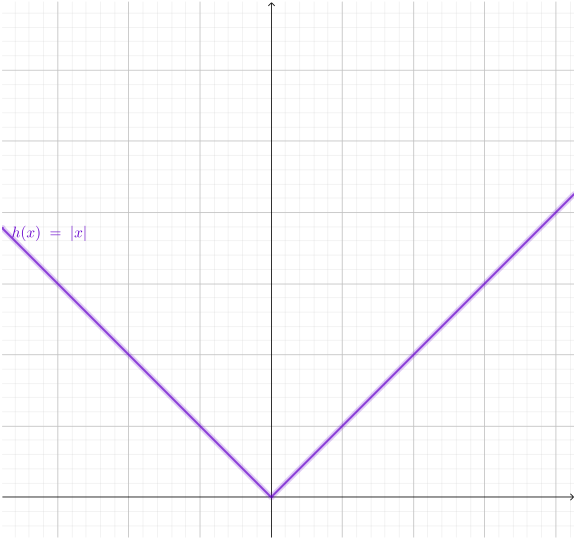
Definition

If the $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists, we denote it be $f'(x_0)$ and we call it the derivative of f at x_0. We say that f is differentiable at x_0. If f is differentiable at any point of an interval I we say that f is differentiable on I.

There are other symbols to denote the derivative of f at x_0:

$$D[f](x_0),\ \frac{df}{dx}(x_0),\ \dot{f}(x_0),\ \frac{df}{dx}_{|x=x_0}, \dots$$

It can not exist



Exercise

Find the tangent line to $y = x^2 + 2x - 1$ in $(2, 7)$.

Interpretation of $f'(x)$

In general $f'(x)$ represents the rate of change of f around x.

Example

If $V(t)$ represent the value of a portfolio at time t then $V'(t)$ represent how that value is changing. In particular if $V'(t) > 0$, then the value is going up.

Increasing/Decreasing functions

- A function $f(x)$ is increasing on an interval I iff for every $x \in I$ we have that $f'(x) \geq 0$
- A function $f(x)$ is strictly increasing on an interval I if for every $x \in I$ we have that $f'(x) > 0$
- A function $f(x)$ is decreasing on an interval I iff for every $x \in I$ we have that $f'(x) \leq 0$.
- A function $f(x)$ is strictly decreasing on an interval I if for every $x \in I$ we have that $f'(x) < 0$.

Derivatives of basic functions

$f(x) =$	$f'(x) =$
c	0
x^q with $q \in \mathbb{Q}$	qx^{q-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$

Attention

You need to know this for the exam!!!

Rules for derivation

- $D[f+g](x) = f'(x) + g'(x)$
- $D[f \cdot g](x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- $D[1/g](x) = -\frac{g'(x)}{(g(x))^2}$
- $D[f/g](x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Chain rule

$$D[f \circ g](x) = g'(x) \cdot (f' \circ g(x))$$

Attention

You need to know this for the exam!!!

Important remark: Several of these rules have additional conditions, such as the derivatives of each of the function involved must exist, or, if you divide by $g(x)$ this cannot be 0...

Questions?

The initial problem

Problem

If the total saving of a country is a function $S(Y)$ of the national product Y, then $S'(Y)$ is called the marginal propensity to save, or MPS. Find the MPS for the following functions:

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Further examples

- Compute the derivative of a^x
- Compute the derivative of $\log_a(x)$
- (part of exam question) Compute the derivative of

$$\sqrt{e^{\sqrt{x^2+1}}}$$

- Compute the derivative of $\ln|x|$, when it exists
- Compute the derivative of $(2x-1)(e^{x^2}+2x-1)^{25}$
- Determine on which interval the function $e^x(x^2-2x+1)$ is increasing/decreasing. Sketch the graph.

Questions?

Higher order derivatives

Observe that the derivative of a function its itself a function and so we can keep on deriving we get

$$f''(x) = \frac{d^2 f}{dx^2}(x)$$
$$f'''(x) = \frac{d^3 f}{dx^3}(x)$$

In general we have

$$f^{(n)}(x) = \frac{d^n f}{dx^n}$$

Thank you for your attention!