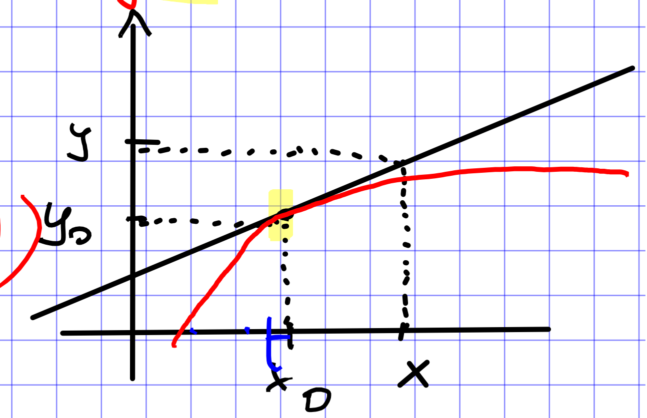
Lecture 4 The story so far

Linear functions $\to$ Slope a & point (x_0, y_0)

$$a = \frac{y - y_0}{x - x_0} \qquad [\, y = a(x - x_0) + y_0 \,]$$



Tangent line to graph of f at $(x_0, f(x_0))$

$$y = f'(x_0)(x - x_0) + f(x_0)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x_0)}{h}$$

($\frac{f(x+h) - f(x)}{h}$ gets closer to $f'(x)$ as h gets closer to 0)

i) $(f+g)'(x) = f'(x) + g'(x)$

ii) $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

If $f(x) = x^r$ $\quad f'(x) = r x^{r-1}$

We learnt how to calculate derivatives of Polynomials

Today

- Rational functions
- Power functions
- Exponential functions
- Logarithms

Properties

i) $(f+g)'(x) = f'(x) + g'(x)$

ii) $(f \cdot g)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

iii) If $g'(x) \neq 0$ & $g(x) \neq 0$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$f(x)$	$f'(x)$
x^r	rx^{r-1}
A	0
(cte)	

$$\frac{d}{dx}(A \cdot f(x)) = A f'(x)$$

Examples.

$$F(x) = \frac{1}{x} = x^{-1} \qquad x \neq 0$$

a) $F'(x) = \frac{d}{dx}(x^{-1}) = (-1)\,x^{-2} = \frac{-1}{x^2}$

b) If we take $[f(x) = 1]$ and $[g(x) = \overset{x^1}{x}]$, we have $F(x) = \frac{f(x)}{g(x)}$

Note that $\boxed{f'(x) = 0}$, and $g'(x) = 1 \cdot x^0 = 1$

So $\left(f/g\right)' = \frac{f' \cdot g - f \cdot g'}{g^2} = \frac{0 \cdot x - 1 \cdot 1}{x^2} = -\frac{1}{x^2}$

ii) $F(x) = \dfrac{3x-5}{x-2} = \dfrac{f(x)}{g(x)}$

$F'(x) = \dfrac{f' \cdot g - f \cdot g'}{g^2} = \dfrac{3 \cdot (x-2) - (3x-5) \cdot 1}{(x-2)^2}$

$= \dfrac{3x-6-3x+5}{(x-2)^2} = \dfrac{-1}{(x-2)^2}$

• $\dfrac{d}{dx}(3x-5) = \dfrac{d}{dx}(3x) + \dfrac{d}{dx}(-5)$

$= \underbrace{\dfrac{d}{dx}(3)}_{0} \cdot x + 3 \cdot \dfrac{d}{dx}(x) + 0$

$= 3 \cdot x^0 = 3 \cdot 1 = 3$

• $\dfrac{d}{dx}(x-2) = \dfrac{d}{dx}(x) + \underbrace{\dfrac{d}{dx}(-2)}_{0}$

$(x = x^1)$

$= 1$

$F(x) = 4x^2 + 3x - 1$

$F'(x) = 4 \cdot 2x^1 + 3 \cdot 1 \cdot x^0 + 0$

The Chain rule

Suppose that f depends on the variable u

u depends on the variable x

So we can form a "new" function

$$g(x) = f(u(x))$$

Example

Suppose $f(u) = u^5$ and $u(x) = 1-x$. So $g(x) = (1-x)^5$

So how can we calculate $\frac{dg}{dx}$ if we know

$\frac{d}{du} f$ and $\frac{du}{dx}$

$$\frac{d}{dx}[f(u(x))] = \overbrace{\frac{d}{du} f(u(x))} \cdot \overbrace{\frac{du}{dx}(x)}$$

Example

- $\frac{d}{du} f(u) = 5 \cdot u^4$
- $\frac{d u}{dx}(x) = -1$

$$\frac{d}{dx}((1-x)^5) = 5u^4 \cdot (-1) = 5(1-x)^4(-1) = -5(1-x)^4$$

Calculate i) $\frac{d}{dx}\left[(x-1)^{-1}\right]$ ii) $\frac{d}{dx}\left((x^2+2x)^{2021}\right)$

iii) $\frac{d}{dx}\left[u(x)^2\right]$

$$\boxed{\frac{d}{dx}\left[f(u(x))\right] = \frac{df}{du}\cdot\frac{du}{dx}}$$

iii) Let $f(u) = u^2$ $\left(\rightarrow \frac{df}{du} = 2u\right)$

$$\frac{d}{dx}\left[f(u(x))\right] = \frac{df}{du}\cdot\frac{du}{dx} = 2u \cdot u'(x).$$

i) $\boxed{(x-1)^{-1} = f(u(x))}$ where $\left[f(u) = u^{-1}\right]$ $\frac{df}{du} = (-1)\,u^{-2}$ ($-1-1 = -2$)

$\left[u(x) = x-1\right]$ $\left[\frac{du}{dx} = 1\right]$

The chain rule yields

$$\frac{d}{dx}\left((x-1)^{-1}\right) = \frac{df}{du}\cdot\frac{du}{dx} = -u^{-2}\cdot 1 = \frac{-1}{(x-1)^2}$$

$$\frac{d}{dx}\Big(\underbrace{(x^2+2x)}^{2021} \Big)$$

$$\begin{cases} u(x) = x^2+2x \\ f(u) = u^{2021} \end{cases}$$

$$f(\overbrace{u(x)}^{\text{inner function}}) = u^{2021} = (x^2+2x)^{2021}.$$

$\underbrace{f}_{\text{outer function}}$

- $\dfrac{df}{du} = 2021\, u^{2020}$

- $\dfrac{du}{dx} = 2x+2$

$$\frac{d}{dx}\big(f(u(x))\big) = \frac{df}{du} \cdot \frac{du}{dx}$$

$$= 2021\,(x^2+2x)^{2020} \cdot (2x+2)$$

Exponential functions

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Calculate

1) $\frac{d}{dx}(2^x) = (\ln 2)2^x$

$$\boxed{e^{\ln y} = y}$$

$$\boxed{\ln(2^x) = x(\ln 2)}$$

$2^x = e^{\ln(2^x)} = e^{(\ln 2)x} = e^u$

where $\boxed{u = (\ln 2)x}$

$$\boxed{\begin{array}{l} u = A \cdot x \\ \hline \frac{du}{dx} = A \end{array}}$$

$\frac{d}{dx}(2^x) = \frac{d}{dx}(e^u) =$

$= \frac{d}{du}(e^u) \cdot \frac{du}{dx} = e^u \cdot (\ln 2)$

$= 2^x \cdot (\ln 2)$

2) Let $\boxed{v(x) = e^{u(x)}}$ where $u = u(x)$. Calculate $\frac{d}{dx} v$.

$= f(u)$ $\qquad f(u) = e^u$

Application. We know that

$$\boxed{x = e^{\ln x}}$$ so $v(x) = x$ and $u(x) = \ln x$

We can use this to calculate

$$\frac{d}{dx}(\ln x)$$

$$\left[x = e^{\ln x} = e^{u} \right. = f(u) \quad \text{where } u = \ln x, \quad \left[f(u) = e^u \right] \quad \frac{df}{du} = e^u$$

Differentiating both sides with respect to x (w.r.t.)

$$1 = \frac{d}{dx}(x) = \frac{d}{dx}\left[f(u(x)) \right] \underset{\text{(chain rule)}}{=} \frac{df}{du} \cdot \frac{du}{dx} = e^u \frac{d}{dx}(\ln x)$$

$$= x \cdot \frac{d}{dx}(\ln x)$$

Hence $\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$ $(x > 0)$

$$\boxed{\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \quad \text{for } x \neq 0}$$

The symbol $|x|$ denotes the absolute value of x which is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example

- $|3| = |-3| = 3 = -(-3)$
- $|7| = |-7| = 7 = -(-7)$

Example

Let $y(x) = x^2 \ln x$, which is defined for $\underline{x > 0}$.

1) Calculate $y'(x)$

2) Study the monotonicity of y

(i.e. determine where is y increasing / decreasing)

Recall

$f(x) = \frac{1}{2}x^2 + 1 \quad // \quad f'(x) =$

Theorem

If $f'(x) \geq 0$ on a whole interval $I \iff f$ is increasing on I

If $f'(x) \leq 0$ on a whole interval $I \iff f$ is decreasing on I

$$y(x) = x^2 \ln x$$
$$= f(x) \cdot g(x)$$

$[f(x) = x^2] \to f'(x) = 2x$

$[g(x) = \ln x] \to g'(x) = \frac{1}{x}$

1) $\frac{d}{dx}(x^r) = r\, x^{r-1}$

2) $\frac{d}{dx}(\ln x) = \frac{1}{x}$

3) $(f \cdot g)' = f' g + f \cdot g'$

So

$$y'(x) = f' \cdot g + f \cdot g' = (2x) \cdot \ln x + x^2 \cdot \left(\frac{1}{x}\right) = 2x \ln x + x$$
$$= x(2\ln x + 1)$$

we want to study for which $x > 0$, $y'(x) > 0$ and $y'(x) < 0$.

First step: Find those $x > 0$ such that $y'(x) = 0$

$y'(x) = 0 \iff x(2\ln x + 1) = 0 \iff \cancel{x = 0 \text{ or }}$ $2\ln x + 1 = 0$

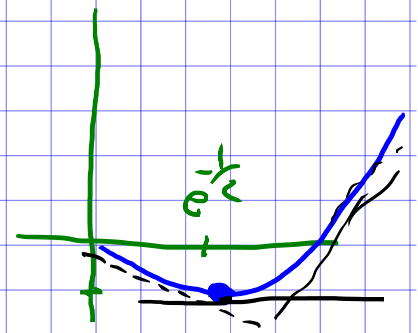
$2\ln x = -1$

$\iff [\ln x = -\frac{1}{2}] \iff e^{-1/2} = e^{\ln x} = x \iff x = e^{-1/2}$

$y(x) = x^2 \ln x$

$y'(x) = x(2\ln x + 1)$

x	$0 < x < e^{-\frac{1}{2}}$	$e^{-\frac{1}{2}}$	$x > e^{-\frac{1}{2}}$
y'	$-$	0	$+$
y	$\searrow$	$y(e^{-\frac{1}{2}})$	$\nearrow$



note that $1 > e^{-\frac{1}{2}}$

and $y'(1) = 1 \cdot (2\underbrace{\ln 1}_{0} + 1) = 1 > 0$

Implicit differentiation

Recall: We have computed (implicitly) $u'(x)$

$x = e^{u(x)}$ $$\left[\frac{d}{dx}\left(e^{u(x)}\right) = e^{u(x)} \cdot u'(x).\right]$$

We can do the same for more "complex" expression

Assume that $u(x)$ is a function satisfying that

$$(*) \quad \boxed{u^2 + xu = 1} \quad \text{and} \quad u(0) = 1.$$

Calculate $u'(0)$

We can differentiate both sides of $(*)$ to obtain

$$\underline{0} = \frac{d}{dx}(1) = \frac{d}{dx}(u^2 + xu) = \frac{d}{dx}(u^2) + \frac{d}{dx}(x \cdot u)$$

$$= 2 \cdot u \cdot u' + \overbrace{\frac{d}{dx}(x)}^{1} \cdot u + x \cdot u' = \underline{2u \cdot u' + u + x \cdot u'}$$

Chain rule. $\left[\begin{array}{l} f(u) = u^2 \\ u = u(x) \\ \frac{d}{dx}[f(u)] = \frac{df}{du} \cdot \frac{du}{dx} \end{array}\right]$

$$\left[(f \cdot g)' = f' \cdot g + f \cdot g'\right]$$

We have that

$$2u\cdot u' + x\cdot u' + u = 0 \quad \& \quad \underline{u(0)=1}$$

then if $x=0$

$$2\cdot u(0)\cdot u'(0) + \overbrace{0\cdot u'(0)}^{0} + u(0) = 0$$

$$\underset{u(0)=1}{\Longleftrightarrow} \quad 2u'(0) + 1 = 0 \iff \boxed{u'(0) = -\tfrac{1}{2}}.$$

So we don't have an explicit expression for $u(x)$

BUT

we know that

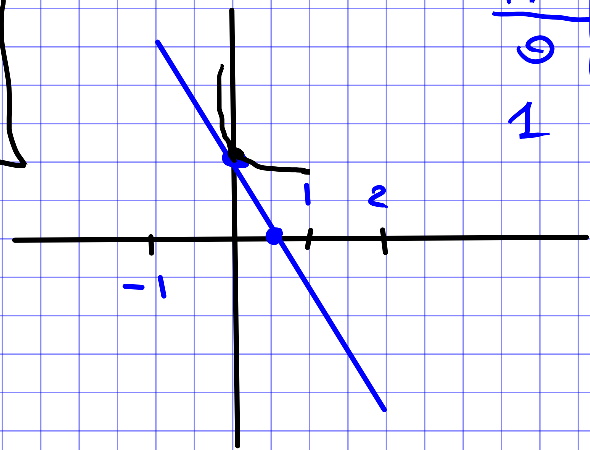
$$u(0) = 1 \quad \& \quad u'(0) = -\tfrac{1}{2}.$$

So the <u>tangent line</u> to the graph of u at $(0, u(0))$

is

$$\boxed{\begin{aligned} Y &= u'(0)(x-0) + u(0) \\ &= -\tfrac{1}{2}x + 1. \end{aligned}}$$

x	$Y(x)$
0	1
1	$\frac{1}{2}$



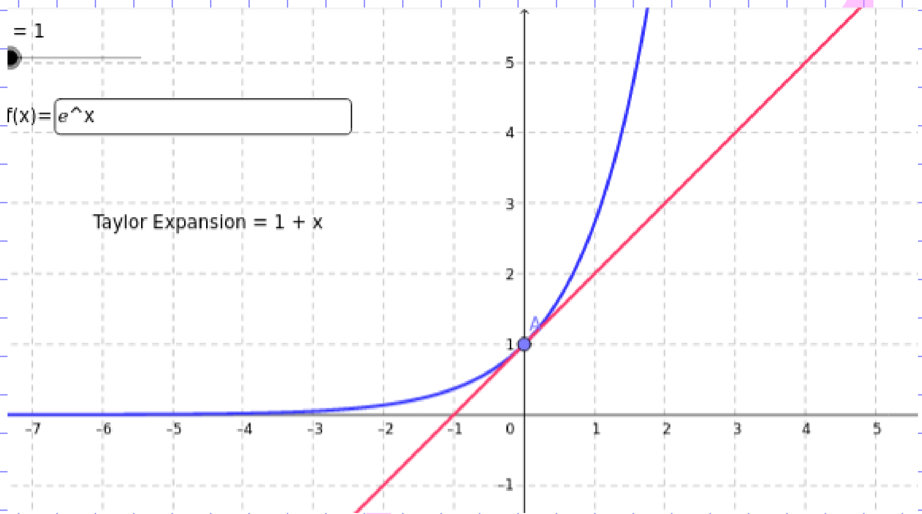
So we can use the tangent line to approximate $u(x)$ for x close to 0

We can then "predict" that $u(\frac{1}{4}) \approx Y(\frac{1}{4}) = -\frac{1}{2}(\frac{1}{4})+1$

$$= 1 - \frac{1}{8} = \frac{7}{8})$$

Polynomial approximation of a function.

(Taylor polynomial of order N about the point x_0



Given a function f the tangent line at a point x_0 is given by the equation

$$\star \quad \boxed{Y = f'(x_0)(x-x_0) + f(x_0)}$$

Example $f(x) = e^x$

$f'(x) = e^x$

If $x_0 = 0$ then $f(0) = f'(0) = 1$

so $\boxed{Y(x) = x + 1}$

The tangent line approximates the values of $f(x)$ for x near zero.

$\star$ is the Taylor polynomial of order 1 about the point x_0

One can expect better approximation is one consider polynomials of higher degree

We define the Taylor polynomial of order 2 about the point x_0 as

$$P_2(x) = \frac{f''(x_0)}{2}(x-x_0)^2 + f'(x_0)(x-x_0) + f(x_0)$$

Note $f''(x)$ is the derivative of the derivative (second derivative)

Example

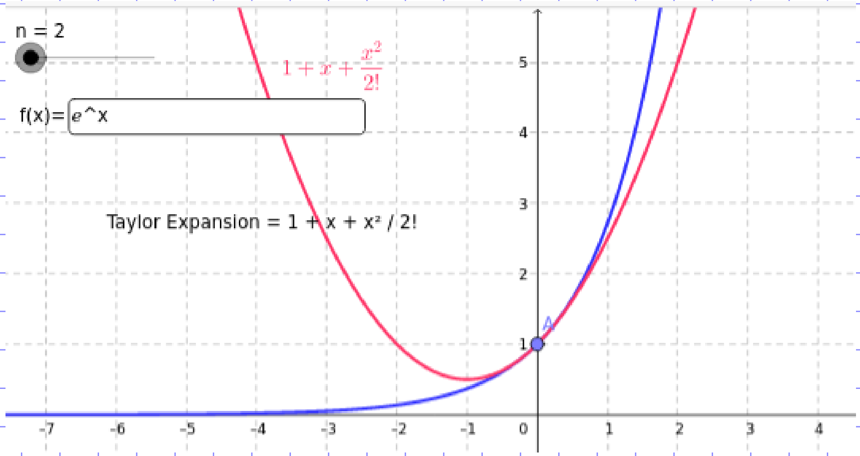
i) $f(x) = e^x = f'(x) = (f')'(x) = f''(x)$

ii) $f(x) = 2x^2 + 3x + 1$

$f'(x) = 4x + 3 \qquad f''(x) = 4 \qquad (f'''(x) = 0)$

For $f(x) = e^x$ and $x_0 = 0$ then

$$P_2(x) = \frac{1}{2}x^2 + x + 1$$



$$P_2(x) = \frac{1}{2}x^2 + x + 1$$

One can see that it approximates better the function $f(x) = e^x$ near $x = 0$

In general: The Taylor polynomial of order n about x_0 is defined as

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$+ \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

where

$$n! = n \cdot (n-1)(n-2) \dots 2 \cdot 1 \quad (n \text{ factorial})$$

and

$f^{(k)}(x_0)$ denotes the kth derivative of f at x_0.

For the exponential function and $x_0 = 0$
you can inspect the following link.
What happens as you increase the order?