

Course MM1005

Lecture 5 : Implicit derivations and Taylor formula

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Questions?

Lecture Goal and Outcome

Goal: Overview of implicit derivation and polynomial approximation.

Learning Outcome: At the end of today lecture you will be able to

solve a problem like the following:

Problem

Suppose that a commodity is sold at price P with a taxation of τ . Suppose that the demand and supply functions are linear, that is

$$D = a - b(P + \tau), \quad S = \alpha + \beta P,$$

Then the equilibrium price is determined by equating the supply to the demand:

$$a - b(P + \tau) = \alpha + \beta P$$

Compute $\frac{dP}{d\tau}$ What is its sign? What is the sign of $\frac{d}{d\tau}(P + \tau)$

Outcome 2

After today you will understand the **rule of 70**:

Rule of 70

Given an interest rate of $p\%$ the doubling time is **approximately** $70/p$ years.

Why you should care

- **Implicit derivation** is a really important technique in economics, as sometimes it is impossible to define a quantity Y directly as a function of another. But still one would want to examine how the two quantities are correlated.
- As you are going to deal with real data and real functions, it will be important to **approximate** them, otherwise some problems cannot be solved.
- Whenever you approximate something it is really important to have an idea of the magnitude of your error. **Taylor** formula provide just that for the approximation we will learn today.

Lecture Plan

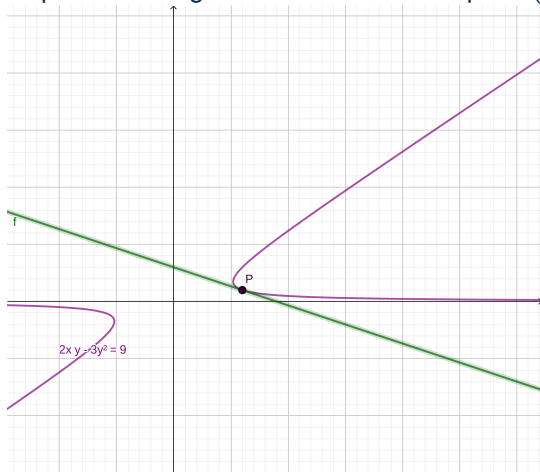
- Implicit derivation (7.1) - usually an exam question
- Linear approximation (7.4)
- Polynomial approximation (7.5)- ALWAYS an exam question.
- Taylor Formula (7.6)

Section 1

Implicit Derivation

Example

Consider the plane curve $2xy - 3y^2 = 9$. We want to compute the slope of the tangent to the curve in the point $(6, 1)$.



The general method

We have an equation $f(x, y) = 0$ and consider $y = y(x)$ we want to determine $y'(x)$ in a point (x_0, y_0) .

- We differentiate each side of $f(x, y) = 0$ with respect to x , considering y as a function of x . We get

$$\frac{d}{dx}f(x, y(x)) = 0,$$

- We set $x = x_0$ and $y = y_0$ and we get an equation in which the only unknown is $y'(x_0)$
- We solve the equation.

If we want to compute $y''(x)$, we repeat the process:

- We compute

$$\frac{d^2}{dx^2}f(x, y(x)) = 0$$

- We set in $x = x_0$, $y = y_0$ and $y'(x) = y'(x_0)$ which we have previously computed.
- We solve for $y''(x_0)$

The initial problem

Problem

Suppose that a commodity is sold at price P with a taxation of τ . Suppose that the demand and supply functions are linear, that is

$$D = a - b(P + \tau), \quad S = \alpha + \beta P,$$

Then the equilibrium price is determined by equating the supply to the demand:

$$a - b(P + \tau) = \alpha + \beta P$$

Compute $\frac{dP}{d\tau}$ What is its sign? What is the sign of $\frac{d}{d\tau}(P + \tau)$

Further Examples

Determine the slope of the tangent of the unit circle at any point (but not $(1,0)$ and $(-1,0)$)

Further Examples

From a previous final

Consider the following curve

$$y^3 e^x + x^2 e^y + y^3 - 2y = C,$$

for some real number C .

- 1 Suppose that the curves passes for $(1, 1)$, determine C ,
- 2 Determine the slope of the curve (that is the slope of its tangent) in the point $(1, 1)$.
- 3 Consider y as a function of x . Is this increasing around $x = 1$?
- 4 Determine the equation of the tangent line in the point $(1, 1)$

Further Examples

From an old exam

Consider the curve

$$y^2 x^2 + \frac{x}{\sqrt{y}} = 6$$

- 1 Compute the slope of the curve in $(2, 1)$.
- 2 Consider y as a function of x , compute $y''(2)$.

Questions?

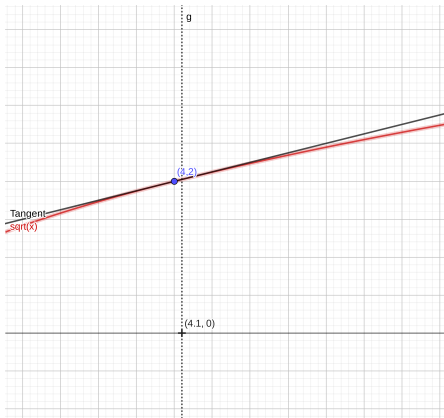
Section 2

Linear approximation

Example

We want to approximate $\sqrt{4,1}$.

Idea: We find $y = t(x)$, the equation of the tangent to the graph of $y = \sqrt{x}$ in the point $x = 4$. Then $t(4,1)$ approximates $\sqrt{4,1}$.



In general

Let $y = f(x)$ the graph of a function differentiable in the point $x = a$.
Then the equation of the tangent to the graph in $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a),$$

When x is near a we have that

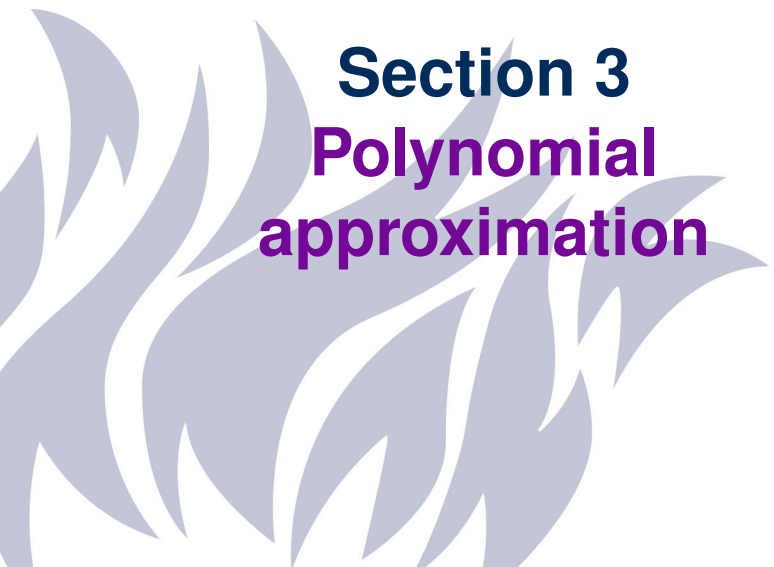
$$f(x) \simeq f(a) + f'(a)(x - a)$$

The rule of 70

Rule of 70

Given an interest rate of $p\%$ the doubling time is approximately $70/p$ years.

Questions?



Section 3

Polynomial approximation

Quadratic approximation

Instead of approximate a function with a line, we approximate it with a quadratic function. That is we want to find

$$p(x) = A + B(x - a) + C(x - a)^2$$

such that $p(x)$ is really close to $f(x)$ when x is near a . We require the following conditions

- $f(a) = p(a)$
- $f'(a) = p'(a)$
- $f''(a) = p''(a)$

We get a unique solution

$$p(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

Examples

- Find the quadratic approximation of $\sqrt{4,1}$
- Find the quadratic approximation of $\ln(1 + \frac{p}{100})$

Higher-order approximations

The Taylor Polynomial

Given a function f differentiable n times at a point a , we have that the degree n Taylor polynomial associated to f in a is:

$$p(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Examples

- Compute the Taylor polynomial of degree 3 of $f(x) = e^x$ around $x = 1$. How do you think the one of degree 50 looks like?
- (Exam question) Compute $p(x)$ the Taylor polynomial of degree 3 of the function $f(x) = x^2 e^x$ around $x = 1$. What is the value of $p(1,1)$?
- (Exam question) Compute $p(x)$ the Taylor polynomial of degree 3 of the function $f(x) = \ln(1 + e^x)$ around $x = 0$. What is the value of $p(0,1)$?
- (Exam question) Compute $p(x)$ the Taylor polynomial of degree 3 of the function $f(x) = \ln(x^2 + 2x + 1)$ around $x = 1$.

Questions?



Section 4

Taylor Formula

Question

How good are our approximations?

Taylor Formula

Suppose that f is a function differentiable $n + 1$ times in an interval I containing the point a . Let $p(x)$ be the Taylor polynomial of degree n of $f(x)$ in a . For every $x \in I$ there is a $z \in I$ such that

$$|f(x) - p(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} (x - a)^{n+1} \right|$$

Example

Compute $e^{1/10}$ with two correct decimals. Suppose that $p(x)$ is the Taylor polynomial of degree n centered in 0 of e^x . We want to compute $p(0,1)$ in such a way that

$$|p(0,1) - e^{0,1}| < 0,005$$

Taylor formula gives us that

$$|p(0,1) - e^{0,1}| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} (0,1)^{n+1} \right|$$

for some z in an interval containing 0 and 0,1. Let us take $I = [0, 1]$. We have

$$\left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| = \left| \frac{e^z}{(n+1)!} (0,1)^{n+1} \right| \leq \frac{e}{10^{n+1}(n+1)!} \leq \frac{3}{10^{n+1}(n+1)!}$$

Questions?

Thank you for your attention!

