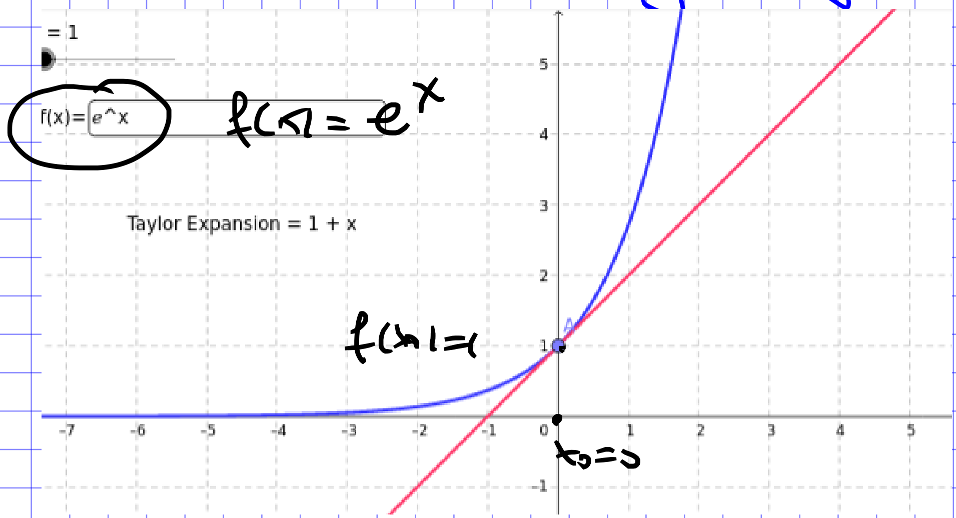
Lecture 5 The story so far



Given a function f the equation of the line tangent to the graph of f at $(x_0, f(x_0))$ is given by

$$\left[\; y = f'(x_0)(x - x_0) + f(x_0) \;\right]$$

We can use this to approximate the value of f near x_0

$$\left[\; y = f'(x_0)(x - x_0) + f(x_0) \;\right]$$

(Taylor polynomial of order 1 about x_0)

One can expect better approximation is one consider polynomials of higher degree

We define the Taylor polynomial of order 2 about the point x_0 as

$$P_2(x) = \frac{f''(x_0)}{2}(x-x_0)^2 + f'(x_0)(x-x_0) + f(x_0)$$

Note: $f''(x)$ is the derivative of the derivative (second derivative)

Example

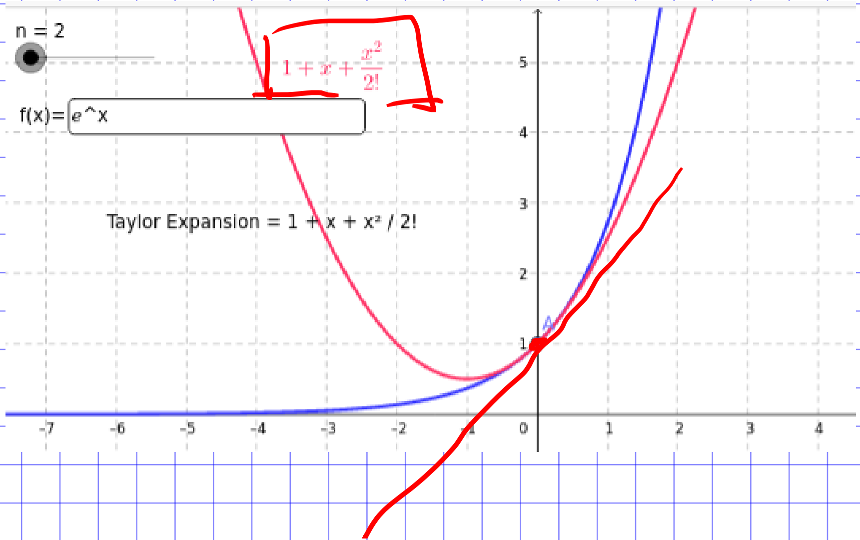
i) $f(x) = e^x = f'(x) = (f')'(x) = f''(x)$

ii) $f(x) = 2x^2 + 3x + 1$

$f'(x) = 4x + 3\cdot 1 \qquad f''(x) = 4 \qquad (f'''(x) = 0)$

For $f(x) = e^x$ and $x_0 = 0$ then $\quad f(0) = e^0 = 1 = f'(0) = f''(0)$

$$P_2(x) = \frac{1}{2}x^2 + 1\cdot x + 1$$



$$P_2(x) = \frac{1}{2}x^2 + x + 1$$

One can see that it approximates better the function $f(x)=e^x$ near $x=0$

In general: The Taylor polynomial of order n $\boxed{\text{about } x_0}$ is defined as

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

If $x_0 = 0$ MacLaurin polynomial.

where

$n! = n\cdot(n-1)(n-2)\dots 2\cdot 1$ (n factorial)

and

$f^{(k)}(x_0)$ denotes the kth derivative of f at x_0.

For the exponential function and $x_0 = 0$ you can inspect the following link.
What happens as you increase the order?

Note

$$\frac{d}{dx}\left[(x-x_0)^k\right] = k(x-x_0)^{k-1} \Rightarrow \left(\frac{d}{dx}\right)^k \left[(x-x_0)^k\right] = k!$$

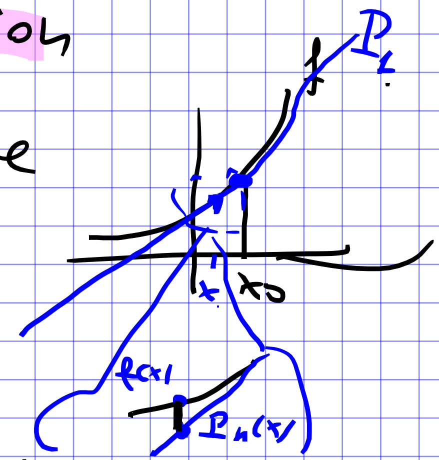
chain rule

$$\left.\begin{aligned} P_n(x_0) &= f(x_0) \\ P_n'(x_0) &= f'(x_0) \\ &\;\;\vdots \\ P_n^{(n)}(x_0) &= f^{(n)}(x_0) \end{aligned}\right\}$$

P_n is a polynomial that coincides with f and all derivatives of order at most n of f at x_0

Taylor's formula

If we define the error of the approximation as $R_{n+1}(x) = f(x) - P_n(x)$. For every x there exists c in between x and x_0 such that



$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

This formula says that

$$f(x) = \overbrace{P_n(x)}^{\text{Taylor Polynomial of order } n} + \overbrace{R_{n+1}(x)}^{\text{(Error term)}}$$

and we have a way to "control" the error

Example

$f(x) = e^x \quad f^{(k)}(x) = e^x \quad f^{(k)}(0) = e^0 = 1$

$$e^x = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + R_4(x)$$

We know that

$3! = 3\cdot 2\cdot 1 = 6$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + R_4(x)$$

and for every x there exist c (in between 0 and x) such that

$$R_4(x) = \frac{e^c}{4!}x^4$$

Thus

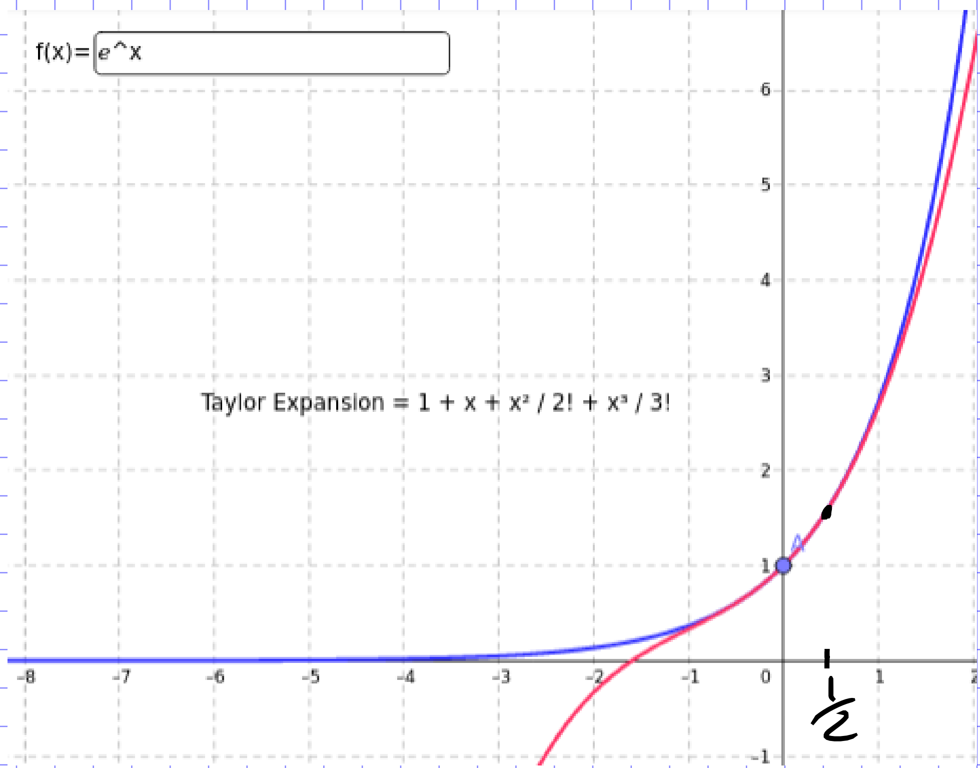
$$\sqrt{e} = e^{1/2} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + R_4(1/2)\left(\frac{1}{2}\right)^4$$

$$\left[R_4(1/2) = \frac{e^c}{24} \cdot \frac{1}{16} \right] \quad \text{with} \quad [0 < c < 1/2 < 1]$$

Therefore, since $e^c \le e \le 3$

$$0 \le R_4\left(\tfrac{1}{2}\right) \le \frac{3}{24 \cdot 16} = \frac{1}{8 \cdot 16} = \frac{1}{128} = 0.0078$$

$|x_{real} - x_{predicted}| < 10^{-}$



$x_{real} = x_{predicted} \pm error$

$error = 10^{-2}$

$1 = 1.01 \pm 0.05$

$[1 = 1.01 + 0.01]$

$$P_1(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0)$$

$$P_1(x_0) = f(x_0)$$

$$P_2(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$$P_3(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

If $x_0 = 0$, Maclaurin's polynomial.

$$\begin{cases} P_1(x) = f(0) + f'(0)x \\ P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \end{cases}$$

If $x_0 = 1$

$$\boxed{P_1(x) = f(1) + f'(1)(x-1)}$$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2}$$

$$(x-1)^2 \cdot \frac{f''(1)}{2}$$

$$\left.\begin{array}{l} f(x) = e^x \\ f'(x) = e^x = f''(x) = f'''(x) \end{array}\right\} f^{(k)}(1) = e^1 = e$$

Continuous functions

We say that a function f is continuous at a point a

if

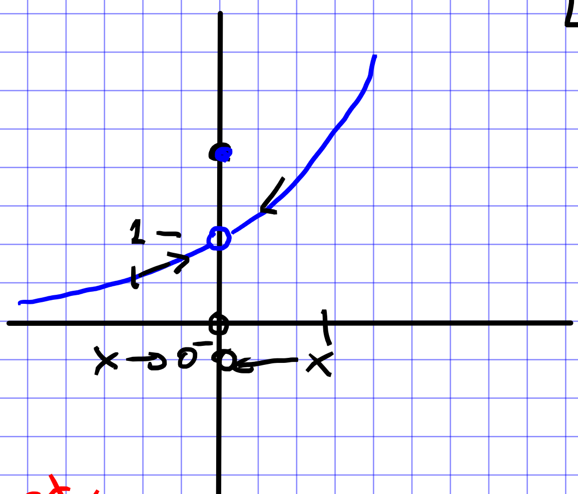
i) f is defined at a

ii) There exists $L = \lim_{x \to a} f(x)$

iii) $L = f(a)$

Examples of not continuous functions

i) The function $f_1(x) = \frac{e^x - 1}{x}$ is not defined at 0



but $\lim_{x \to 0} f_1(x) = 1$

Lets define $f_2(x) = \begin{cases} \frac{e^x - 1}{x} & x \neq 0 \\ 2 & x = 0 \end{cases}$

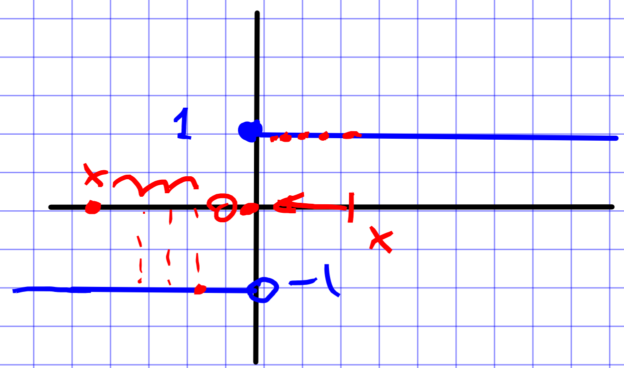
- f_2 is defined at 0
- $\lim_{x \to 0} f(x) = 1$ but $f(0) \neq 1$

$f_3(x) = \begin{cases} \frac{e^x - 1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

f_3 is continuous at 0

ii) The function $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ is defined at 0

but $\lim_{x \to 0} f(x)$ does not exists



If x approaches 0 from the right hand side, f(x) gets closer to 1

but

If x approaches 0 from the left f(x) gets closer to -1.

That is

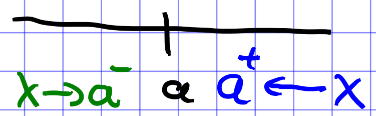
$$\lim_{x \to 0^+} f(x) = 1 \qquad \lim_{x \to 0^-} f(x) = -1.$$

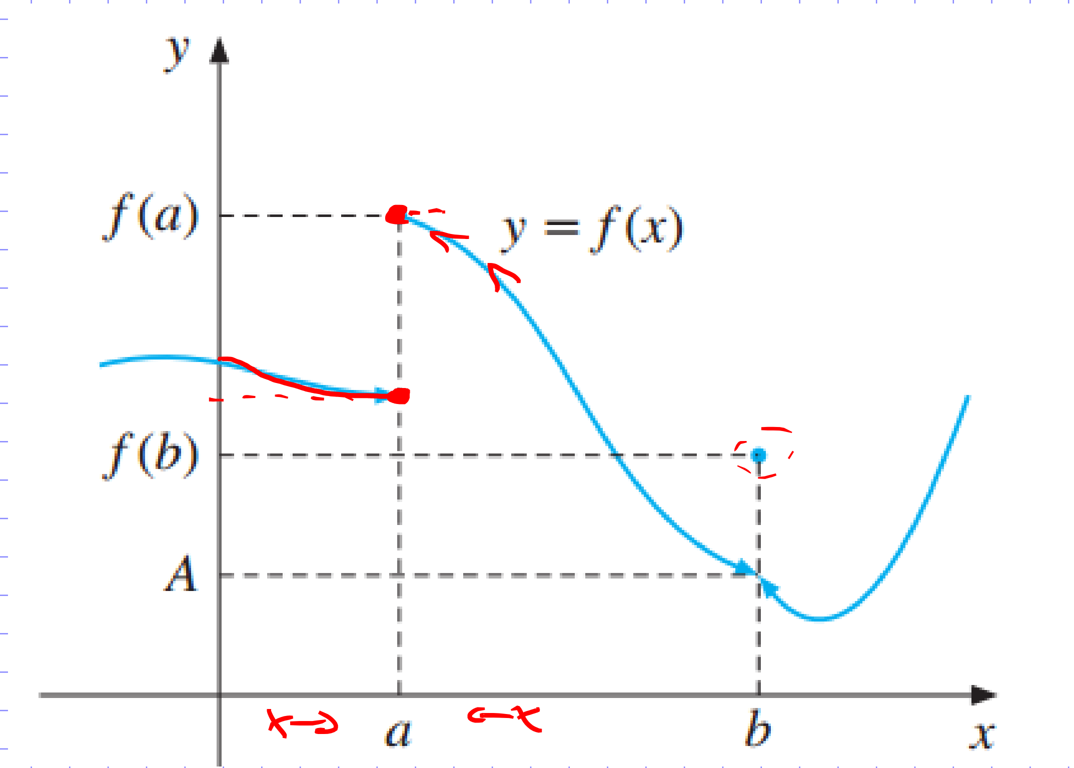
$\lim_{x \to a} f(x)$ exists if both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exists and coincide

We define

$\lim_{x \to a^+} f(x)$ as the limit when x approaches a from above

$\lim_{x \to a^-} f(x)$ as the limit when x approaches a from below





If f and g are continuous at a, then

(a) $f + g$ and $f - g$ are continuous at a

(b) fg and f/g (if $g(a) \neq 0$) are continuous at a

(c) $[f(x)]^r$ is continuous at a if $[f(a)]^r$ is defined (r any real number)

(d) If f is continuous and has an inverse on the interval I, then its inverse f^{-1} is continuous on $f(I)$.

Any function that can be constructed from continuous functions by combining one or more operations of addition, subtraction, multiplication, division (except by zero), and composition, is continuous at all points where it is defined.

Example

$$f(x) = \frac{x^3+2x+3}{(x-1)(x-2)}$$

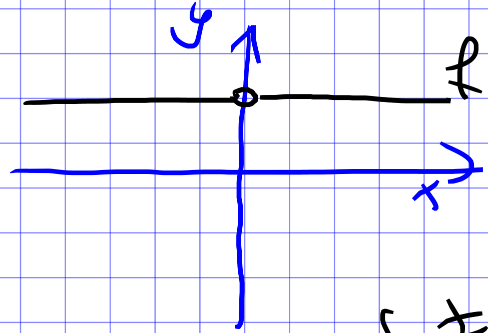
Determine at which values the function f is continuous : f is continuous at every point x except for $x=1$ and $x=2$.

• Polynomials are continuous

$f(x)=x$ 1) f is defined everywhere 2) $\lim_{x\to a} f(x) = a = f(a)$

Q: What happens as x gets closer to 1?

$f(x) = \frac{x}{x}$ is defined for all $x \neq 0$



For all $x \neq 0$ $\quad f(x) = \frac{x}{x} = 1$

$\left(\frac{e^x - 1}{x} \right)$ $\qquad \frac{(x-1)(x-2)}{(x-1)} = x-2$ for $x \neq 1$.

$f_2(x) = \begin{cases} \frac{x}{x} & x \neq 0 \\ 2 & x = 0 \end{cases}$

Example Let $f(x) = \dfrac{1}{(x+2)^2}$ [Is not defined for $x=-2$]

Table 1 Values of $1/(x+2)^2$ when x is close to -2

x	-1.8	-1.9	-1.99	-1.999	-2.0	-2.001	-2.01	-2.1	-2.2
$\dfrac{1}{(x+2)^2}$	25	100	10000	1000000	*	1000000	10000	100	25

*not defined

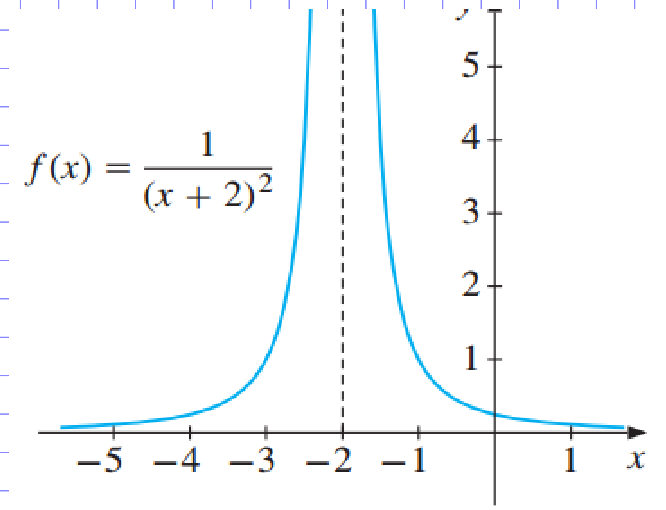


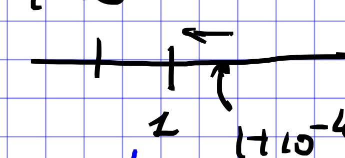
Figure 1 $f(x) \to \infty$ as $x \to -2$

We say that f has a vertical asymptote at $x=-2$ $\left(\lim_{x\to -2} f(x) = +\infty\right)$

Let $f(x) = \dfrac{1}{x-1}$. Calculate $\lim_{x\to 1} f(x)$ & $\lim_{x\to 1} f(x)$

$0,9999\ldots9$

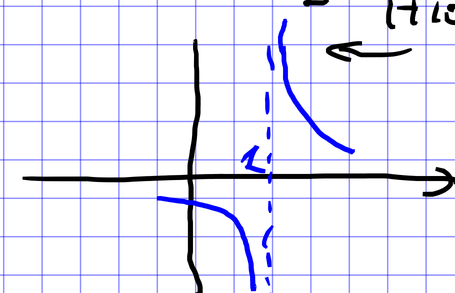
$1-10^{-4}$



$x = 1+10^{-k}$ $\quad k=1,2,3,\ldots$ $\quad x = 1+10^{-1} = 1+0.1$

$x = 1+0.01$

$x = 1+0.001$

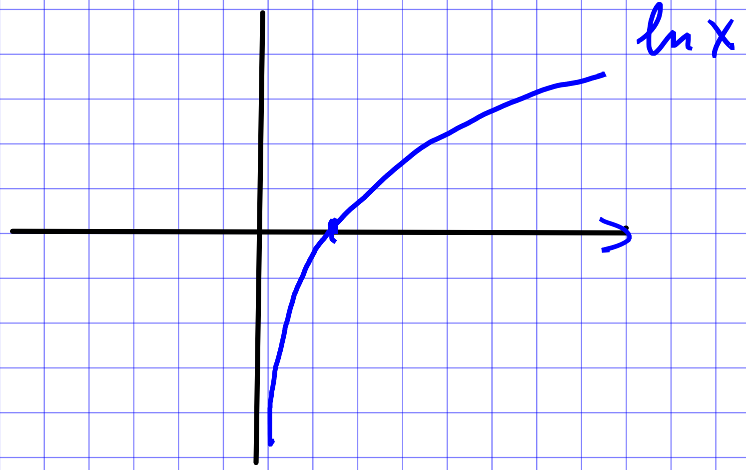


$f(1+10^{-k}) = \dfrac{1}{1+10^{-k}-1} = \dfrac{1}{10^{-k}} = (10^{-k})^{-1} = 10^{k}$

$f(1-10^{-k}) = \dfrac{1}{-10^{-k}} = (-1)(10^{-k})^{-1} = -10^{k}$

$$\lim_{x\to 1^+} \frac{1}{x-1} = +\infty \quad \& \quad \lim_{x\to 1^-} \frac{1}{x-1} = -\infty$$

$\frac{1}{x-1}$ has a vertical asymptote at $x=1$.



$f(x) = \ln x$

- Is defined only for $x > 0$
- $\lim_{x\to 0^+} f(x) = -\infty$

$x = 10^{-k}$ $\quad k = 1, 2, 3, \dots$

$x = 0,000 \dots 1$

So $\ln(10^{-k}) = (-k)\,\underline{\ln 10}$

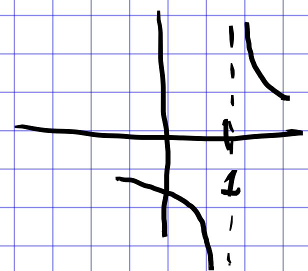
f has a vertical asymptote at $x=0$

$$f(x) = \frac{x^3+2x+3}{(x-1)(x-2)}$$ What happens as $x \to 1$?

$\lim\limits_{x\to 1^+} f(x)$? $\qquad \lim\limits_{x\to 1^-} f(x)$?

$$f(x) = \left(\frac{x^3+2x+3}{x-2}\right)\frac{1}{x-1} = g(x)\,\frac{1}{x-1}$$

- $$\lim_{x\to 1} g(x) = \lim_{x\to 1}\frac{x^3+2x+3}{x-2} = \frac{\lim\limits_{x\to 1}(x^3+2x+3)}{\lim\limits_{x\to 1}(x-2)}$$
$$= \frac{1^3+2\cdot 1+3}{1-2} = \frac{6}{-1} = -6$$

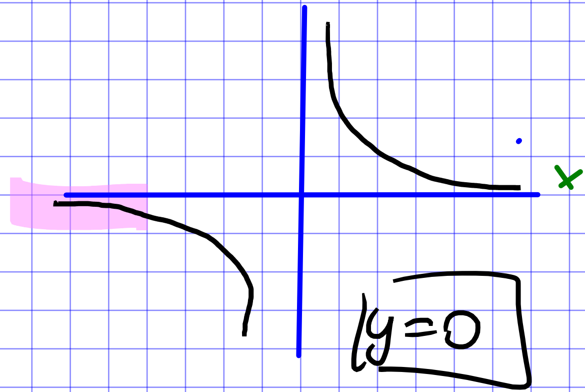


- $$\lim_{x\to 1^+}\frac{1}{x-1} = +\infty$$
- So we have that $\lim\limits_{x\to 1^+} f(x) = \left[(-6)\cdot(+\infty)\right]$
$$= -\infty$$
- $$\lim_{x\to 1^-}\frac{1}{x-1} = -\infty \to \lim_{x\to 1^-} f(x) = \left[(-6)\cdot(-\infty)\right] = +\infty$$

Long time behaviour of functions

What happens with $\left[f(x) = \frac{1}{x} \right]$ as x gets larger and larger?

x	10	100	...	10^7	......
$f(x)$	0,1	0,01		0'0000001	



We say that the function f has limit L as x tends to $+\infty$ (or $-\infty$) if we can make $f(x)$ arbitrarily close to L by making x sufficiently large. We write

$$L = \lim_{x \to +\infty} f(x)$$

We say that the horizontal line $y = L$ is a horizontal asymptote for f

Warning In some situations we need more information to determine a limits as we can run into an *indeterminate form*

$$\left[\frac{0}{0}\right] \quad \left[\frac{\infty}{\infty}\right] \quad [0 \times \infty] \quad [\infty - \infty] \quad [0^0] \quad [1^\infty] \quad [\infty^0]$$

Examples of $\frac{\infty}{\infty}$

Calculate $\lim_{x \to \infty} f(x)$ where

i) $f(x) = \frac{x+1}{x}$ ii) $f(x) = \frac{x^2+1}{x}$ iii) $f(x) = \frac{x}{x^2+1}$

L'HÔPITAL'S RULE $\left[\frac{0}{0}\right]$ or $\left[\frac{\infty}{\infty}\right]$

Suppose that f and g are differentiable in an interval (α, β) that contains a, except possibly at a, and suppose that $f(x)$ and $g(x)$ both tend to 0 as x tends to a. If $g'(x) \neq 0$ for all $x \neq a$ in (α, β), and if $\lim_{x\to a} f'(x)/g'(x) = L$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L$$

This is true whether L is finite, ∞, or $-\infty$.

Note: $x \to a$ can be replaced by $x \to a^-$, $x \to a^+$, $x \to +\infty$, $x \to -\infty$

It also works for indeterminate form $\left[\frac{\infty}{\infty}\right]$

Examples

i) $\lim_{x\to 0} \frac{e^x - 1}{x}$

ii) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

iii) $\lim_{x \to 0^+} x \ln x \qquad (0 \cdot \infty)$ Note: $\lim_{x \to 0^+} x = 0$

$$\left[\lim_{x \to 0^+} \ln x = -\infty \right]$$

iv) $\lim_{x \to \infty} \left(\sqrt{x^2 + x} - x \right) \qquad (\infty - \infty)$

v) $\lim\limits_{x \to \infty} \dfrac{x^p}{e^x} \qquad p > 0$

$$\ln\left(\frac{x^p}{e^x}\right) = \ln(x^p) - \ln e^x = p\ln x - x$$
$$= x\left(p\,\frac{\ln x}{x} - 1\right)$$

$$\lim_{x \to \infty} \frac{\ln x}{x} = \left[\frac{\infty}{\infty}\right] \underset{\text{L'Hôpital's}}{=}$$

So $\lim\limits_{x \to +\infty} \left(p\,\dfrac{\ln x}{x} - 1\right) = -1$. Hence

$$\lim_{x \to +\infty} \frac{x^p}{e^x} = \left[a = e^{\ln a}\right] = \lim_{x \to +\infty} e^{\ln\left(\frac{x^p}{e^x}\right)} =$$
$$= e^{\lim\limits_{x \to +\infty} \ln\left(\frac{x^p}{e^x}\right)} = \left[e^{-\infty}\right] = 0$$