Course MM1005

Lecture 7: Optimization and graphing

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Questions?

Lecture Goal and Outcome

Goal: Understand how the graph of a function looks like by looking at its law. Solve optimization problems.

Learning Outcome: At the end of today lecture you will be able to solve problem like the following

Problem

A firm production function is $Q(L) = 12L^2 - \frac{1}{20}L^3$, where $L \in [0, 200]$ denotes the number of workers.

1. What number of worker maximise Q?
2. Let L^* denote the size of the workforce that maximizes the output per worker $Q(L)/L$. Find L^*.

Why you should care

- Optimization is the heart of Economic analysis. Everyone of your future employer will want to maximize the revenue and minimize the costs.
- Some techniques you have glanced in high school and you will see again in deep in other courses (like in Econometrics) are just optimization problems under disguise. For example: linear regression is an example of an optimization problem in many variables.

Lecture Plan

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- Extreme points (definition and existence) (9.1, 9.4)
- The hunt for critical points (9.2)
- Local extreme points (9.6)
- The second derivative test - Inflection points (8.6)

Attention!

We are going to solve a lot of previous finals exercises

Section 1
Extreme points

Extreme points

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Definition

Let f be a function with domain D, we say that

1. $a \in D$ is a (global) maximum point for f and $f(a)$ is the maximum value of f if
$$f(x) \leq f(a) \text{ for every } x \in D$$
2. $a \in D$ is a (global) minimum point for f and $f(a)$ is the minimum value of f if
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Example

The point $x = 0$ is a minimum point of $x^2 - 30$. The minimum value of the function $f(x) = x^2 - 30$ is $f(0) = -30$.

They exists!

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Extreme value Theorem

A continuous function defined on a closed interval $[a, b]$ has (at least) a maximum and a minimum point.

They exists!

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Extreme value Theorem

A continuous function defined on a closed interval $[a, b]$ has (at least) a maximum and a minimum point.

Example

Find the maximum and the minimum points of $f(x) = x^2 + 5$ on $[-2, 2]$. Give the maximum and the minimum value of f on $[-2, 2]$.

Questions?

Section 2
The hunt for extreme points

Critical points and extreme points

Definition

A point c where a function f is differentiable and such that $f'(c) = 0$ is called critical point for f.

Necessary first order condition

Suppose that the function f, defined on $[a, b]$ is differentiable in (a, b). If $c \in (a, b)$ is a maximum or minimum point for f on $[a, b]$, then c is a critical point for f, that is

$$f'(c) = 0$$

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Consequences

This tell us that min/max point on a closed interval $[a, b]$ can be

- on critical points
- on the extremes a and b
- in points where f is not differentiable

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The initial problem

A firm production function is $Q(L) = 12L^2 - \frac{1}{20}L^3$, where $L \in [0, 200]$ denotes the number of workers.

1. What number of worker maximise Q?
2. Let L^* denote the size of the workforce that maximizes the output per worker $Q(L)/L$. Find L^*.

Questions?

Section 3
Local extreme points

Local extreme points

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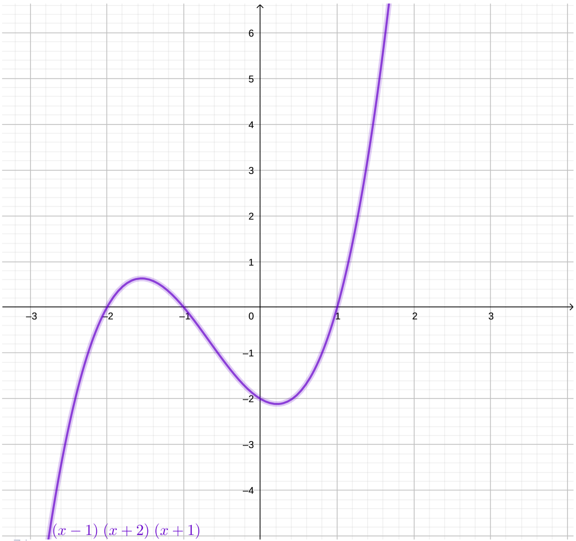
Definition

Let f be a function with domain D, we say that

1. $a \in D$ is a local maximum point for f if there is an interval $I = (\alpha, \beta)$ containing a such that
$$f(x) \leq f(a) \text{ for every } x \in I \cap D$$
2. $a \in D$ is a local minimum point for f if there is an interval $I = (\alpha, \beta)$ containing a such that
$$f(x) \geq f(a) \text{ for every } x \in I \cap D$$

Example



The first derivative test

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Necessary condition

Let f be defined on an interval I and $c$be an element of the interior of I such that f is differentiable at c. If f has a local extreme point at c, then c is a critical point for f.

First derivative test

Consider the function $f(x)$ and let c be a critical point for f

- If $f'(x)$ change sign at c from positive to negative, then $x = c$ is a local

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- If $f'(x)$ change sign at c from positive to negative, then $x = c$ is a local maximum point.
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The first derivative test

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First derivative test

Consider the function $f(x)$ and let c be a critical point for f

- If $f'(x)$ change sign at c from positive to negative, then $x = c$ is a local maximum point.
- If $f'(x)$ change sign at c from negative to positive, then $x = c$ is a local minimum point.
- If $f'(x)$ does not change sign then c is not a local extreme (it could be an inflection point)

Example from an old exam

Consider the function

$$f(x) = \sqrt{x}e^{-x}$$

1. Find the domain of f
2. Find in which interval the function is increasing/decreasing. Find all local extreme point and decide their type.
3. Compute $\lim_{x\to\infty} f(x)$
4. Sketch the graph

Example from an old exam

Stockholm University

Consider the function

$$f(x) = \sqrt{x}e^{-x}$$

1. Find the domain of f
2. Find in which interval the function is increasing/decreasing. Find all local extreme point and decide their type.
3. Compute $\lim_{x\to\infty} f(x)$
4. Sketch the graph (Sofia's Tip: do this along the other points)

Example from an old exam II

Consider the function

$$f(x) = \frac{(4+x)^2}{x-1}$$

1. Find the domain of f
2. Find in which intervals the function is increasing/decreasing. Find all local extreme points and decide their type.
3. Compute $\lim_{x\to\infty} f(x)/2x$
4. Has f and global maximum or minimum?

Example from an old exam III

Consider the function

$$f(x) = x^3(x-1)$$

1. Find in which intervals the function is increasing/decreasing.
2. Find the maximum and minimum value of f in the interval $[-1, 1]$.
3. Compute $\lim_{x\to\infty} f(x)/e^x$,

Section 4
Convexity and second derivatives

Convexity

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We are not going to give the formal definition of convex and concave functions. We will just say, really informally, that a function is convex around a point *a* if its graph looks like a bowl. It is concave if it looks like a cap.

Convexity

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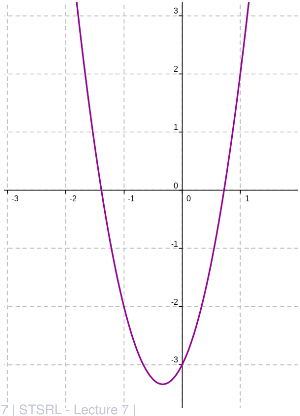

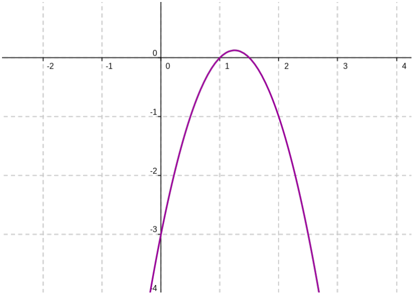

The second derivative tests

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Test for convexity

Let f be a function twice differentiable in an interval I.

1. If $f''(c) \geq 0$ for every point c in I we have that f is convex on I.
2. If $f''(c) \leq 0$ for every point c in I we have that f is concave on I.

Test for local extrema

Let f be a function twice differentiable in an interval I, and let c be a critical point of f, lying in the interior of I.

1. If $f''(c) > 0$ we have that c is a local minimum.
2. If $f''(c) < 0$ we have that c is a local maximum
3. If $f''(c) = 0$ it can be neither or either. In any case we say that c is an inflection point.

Example (Old Exam)

Consider the function

$$f(x) = (x^2 - 9)e^{5x}$$

1. Find the the places where the function is 0, all local extreme points and decide their type. Find where the function is concave and convex.
2. Sketch the graph and compute, if exist the limits of $f(x)$ when x tends to $+\infty$ and $-\infty$.

Questions?

Thank you for your attention!