Warning In some situations we need more information to determine a limits as we can run into an *indeterminate form*

$$\left[\frac{0}{0}\right] \quad \left[\frac{\infty}{\infty}\right] \quad [0\times\infty] \quad [\infty-\infty] \quad [0^0] \quad [1^\infty] \quad [\infty^0]$$

Examples of $\frac{\infty}{\infty}$

Note: For $r>0$

$$\boxed{\lim_{x\to+\infty} x^r = +\infty}$$

Calculate $\lim\limits_{x\to\infty} f(x)$ where

i) $f(x) = \frac{x+1}{x}$ ii) $f(x) = \frac{x^2+1}{x}$ iii) $f(x) = \frac{x}{x^2+1}$

$$\lim_{x\to+\infty} \frac{x+1}{x} = \left[\frac{\infty}{\infty} = \text{"}\frac{\lim\limits_{x\to\infty} x+1}{\lim\limits_{x\to\infty} x}\text{"}\right] = \lim_{x\to+\infty} \frac{x\,(1+1/x)}{x}$$

$$= \lim_{x\to+\infty} (1+1/x) = \lim_{x\to+\infty} 1 + \underbrace{\lim_{x\to+\infty} \frac{1}{x}}_{0} = 1+0 = 1$$

$$\left[\cancel{\lim_{x\to+\infty} \frac{x+1}{x} = \frac{\infty}{\infty} = 1}\right]$$

$$\lim_{x\to+\infty} \frac{x^2+1}{x} = \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to+\infty} \frac{x^2(1+1/x^2)}{x} = \lim_{x\to+\infty} x\cdot(1+1/x^2)$$

$$= [\infty\cdot 1] = \infty$$

iii) $\lim_{x \to +\infty} \frac{x}{x^2+1} = \left[\frac{\infty}{\infty}\right] = \lim_{x \to +\infty} \frac{\cancel{x}}{x^{\cancel{2}}\left(1+\frac{1}{x^2}\right)} =$

$$= \lim_{x \to +\infty} \frac{1}{x\left(1+\frac{1}{x^2}\right)} = \frac{\lim_{x \to +\infty} 1}{\left(\lim_{x \to +\infty} x\right)\left(\lim_{x \to +\infty}\left(1+\frac{1}{x^2}\right)\right)} = \left[\frac{1}{\infty \cdot 1}\right] = 0$$

Note that
- $\lim_{x \to +\infty} 1 = 1$
- $\lim_{x \to +\infty} \left(1+\frac{1}{x^2}\right) = 1 + 0 = 1$
- $\lim_{x \to +\infty} x = +\infty$

L'HÔPITAL'S RULE $\left[\frac{0}{0}\right]$ or $\left[\frac{\infty}{\infty}\right]$

Suppose that f and g are differentiable in an interval (α, β) that contains a, except possibly at a, and suppose that $f(x)$ and $g(x)$ both tend to 0 as x tends to a. If $g'(x) \neq 0$ for all $x \neq a$ in (α, β), and if $\lim_{x\to a} f'(x)/g'(x) = L$, then

$$\left(\left[\frac{0}{0}\right]\right) = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L$$

This is true whether L is finite, ∞, or $-\infty$.

Note: $x \to a$ can be replaced by $x \to a^-$, $x \to a^+$, $x \to +\infty$, $x \to -\infty$

It also works for indeterminate form $\left[\frac{\infty}{\infty}\right]$

Examples

L'Hôpital's rule.

(i) $\lim_{x\to 0} \frac{e^x - 1}{x} = \left[\frac{0}{0}\right] = \lim_{x\to 0} \frac{(e^x-1)'}{(x)'} = \lim_{x\to 0} \frac{e^x}{1} = \frac{e^0}{1} = 1$

$\left[\lim_{x\to a} (A\, f(x)) = A \cdot \lim_{x\to a} f(x)\right]$

L'Hôpital's rule

ii) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \left[\frac{0}{0}\right] = \lim_{x\to 0} \frac{e^x - 1}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{e^x - 1}{x} = \frac{1}{2} \cdot 1$

$= 1/2$.

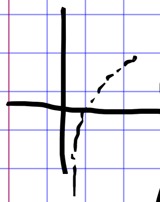
- $\lim_{x \to \infty} \frac{x^{2021}}{x^{2021}} = 1$

- $\lim_{x \to +\infty} \frac{(x+1)^{2021}}{(x+2)^{2021}} = \left[\frac{\infty}{\infty}\right] = \lim_{x \to +\infty} \left(\frac{x+1}{x+2}\right)^{2021}$

$= \left(\lim_{x \to \infty} \frac{x+1}{x+2}\right)^{2021} = \left[\frac{\infty}{\infty}\right]^{2021}$

$= \left(\lim_{x \to \infty} \frac{1+1/x}{1+2/x}\right)^{2021} = 1^{2021} = 1.$

iii) $\lim_{x\to 0^+} x\ln x \qquad [0\cdot\infty] \qquad$ Note: $\lim_{x\to 0^+} x = 0$



$\left(\lim_{x\to 0^+} \ln x = -\infty\right)$

$x = (x^{-1})^{-1} = \frac{1}{x^{-1}} = \frac{1}{\frac{1}{x}}$

$\left[\lim_{x\to 0^+} \ln x = -\infty\right]$

$$\lim_{x\to 0^+} x\ln x = [0\cdot\infty] = \lim_{x\to 0^+}\frac{\ln x}{x^{-1}} = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \left[\frac{\infty}{\infty}\right]$$

$$\underset{\text{L'Hôpital}}{=} \lim_{x\to 0}\frac{\frac{1}{x}}{(-1)x^{-2}} = \lim_{x\to 0}(-1)\frac{x^2}{x} = (-1)\left(\lim_{x\to 0^+} x\right) = 0$$

$$\frac{\frac{1}{x}}{x^{-2}} = \frac{x^{-1}}{x^{-2}} = (x^{-2})^{-1}\cdot x^{-1} = x^2\cdot x^{-1} = \frac{x^2}{x}$$

iv) $\lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right) = [\infty - \infty]$

$$\sqrt{x^2+x} - x = \frac{\left(\sqrt{x^2+x} - x\right)\left(\sqrt{x^2+x} + x\right)}{\sqrt{x^2+x} + x}$$

$[a^2 - b^2 = (a-b)(a+b)]$

$$= \frac{\left(\sqrt{x^2+x}\right)^2 - x^2}{\sqrt{x^2+x} + x} = \frac{x^2 + x - x^2}{\sqrt{x^2+x} + x} = \frac{x}{\sqrt{x^2+x} + x} = \cancel{1}$$

$$\left[\sqrt{x^2+x} = \sqrt{x^2\left(1+\frac{1}{x^2}\right)} = \sqrt{x^2}\,\sqrt{1+\frac{1}{x^2}} = x\sqrt{1+\frac{1}{x^2}}\right]$$

$$* = \frac{x}{x\sqrt{1+\frac{1}{x^2}}+x} = \frac{x}{x\left(\sqrt{1+\frac{1}{x^2}}+1\right)} = \frac{1}{\sqrt{1+\frac{1}{x^2}}+1}.$$

$$\lim_{x\to+\infty}\left(\sqrt{x^2+x}-x\right) = \lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}+1} = \frac{1}{1+1} = \frac{1}{2}$$

$$\left[\frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty}\sqrt{1+\frac{1}{x^2}}+\lim_{x\to\infty} 1}\right]$$

$$\left[\lim_{x\to+\infty}\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}} = \left(\lim_{x\to+\infty}\left(1+\frac{1}{x^2}\right)\right)^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1.\right.$$

v) $\lim\limits_{x \to \infty} \dfrac{x^p}{e^x}$ $\quad p > 0$

$$\ln\left(\frac{x^p}{e^x}\right) = \ln(x^p) - \ln e^x = p \ln x - x$$
$$= x\left(p \frac{\ln x}{x} - 1\right)$$

$$\lim_{x \to \infty} \frac{\ln x}{x} = \left[\frac{\infty}{\infty}\right] \underset{\text{L'Hôpital's}}{=}$$

So $\lim\limits_{x \to +\infty} \left(p \dfrac{\ln x}{x} - 1\right) = -1$. Hence

$$\lim_{x \to +\infty} \frac{x^p}{e^x} = \left[a = e^{\ln a}\right] = \lim_{x \to +\infty} e^{\ln\left(\frac{x^p}{e^x}\right)} =$$
$$= e^{\lim\limits_{x \to +\infty} \ln\left(\frac{x^p}{e^x}\right)} = \left[e^{-\infty}\right] = 0$$

Lecture 6

We say that a function f is continuous at a point a

if

i) f is defined at a

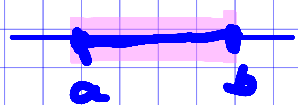
ii) There exists $L = \lim_{x \to a} f(x)$

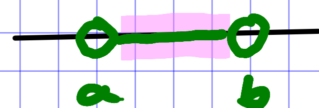
iii) $L = f(a)$

We say that a function is continuous on an interval I if it's continuous at every point of the interval.

$[a, b]$ is the interval of all real values x such that $a \leq x$ and $x \leq b$

(a, b) is the interval of all real values x such that $a < x$ and $x < b$





Notation

$\mathbb{R} = (-\infty, +\infty)$

$\left[\lim_{x \to +\infty} x = +\infty \right]$

Optimisation of functions of one variable

Let $f : I \to \mathbb{R}$, and a a point in I. We say that

f attains a maximum (on I) at a if for all $x \in I$ $f(x) \le f(a)$

$f(a)$ is the maximum of f on I

f attains a minimum (on I) at a if for all $x \in I$ $f(x) \ge f(a)$

$f(a)$ is the minimum of f on I

Example

Consider $f(x) = x^2 - 4x + 3$

Recall

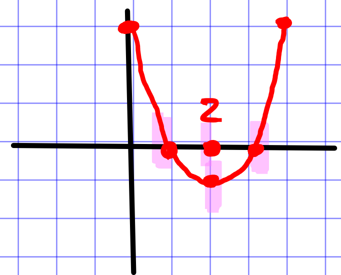
$(x-a)^2 = x^2 - 2 \cdot a x + a^2$

$$= x^2 - 2 \cdot 2x + 3 + 2^2 - 2^2$$

$$= (x-2)^2 - 1.$$

Hence, for all $x \in \mathbb{R}$, since $(x-2)^2 \ge 0$, we have

$$f(x) = (x-2)^2 - 1 \ge 0 - 1 = f(2)$$



So f attains its minimum value at $x = 2$

and $f(2) = -1$ is the minimum of f on $\mathbb{R}$

- We know $(2, f(2)) = (2,-1)$ is the vertex of the parabola
- $f(x) = 0 \iff (x-2)^2 - 1 = 0 \iff (x-2)^2 = 1$

$$\iff x-2 = \pm\sqrt{1} = \pm 1$$

$$\iff x = 2 \pm 1 = \begin{cases} 3 \\ 1 \end{cases}$$

(or use the p-q formula)

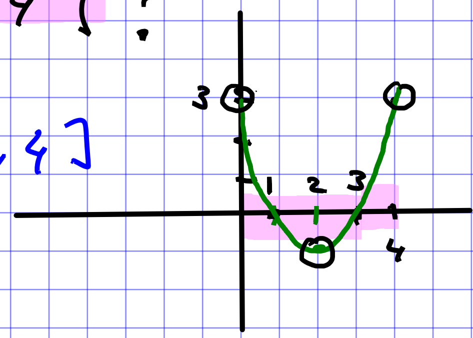
- Since the coefficient of x^2 is positive we know that the parabola $f(x) = x^2 - 4x + 3$ is convex

- $f(0) = 3$
- Note that $\lim_{x\to+\infty} (x^2 - 4x + 3) = +\infty^{(*)}$, so the function has no maximal value on $\mathbb{R}$.

(*) Moral "The winner takes it all... " ("leading term")

$$\lim_{x\to+\infty} (x^2-4x+3) = \lim_{x\to+\infty} x^2\left(1 - \frac{4}{x} + \frac{3}{x^2}\right) = +\infty \cdot 1 = +\infty$$

What happens if we change the domain to be $[0,4]$?



Now the function attains its maximum value on $[0,4]$ at $x=0$ and $x=4$

Moral

1) The solution of the optimisation problem depends on the domain

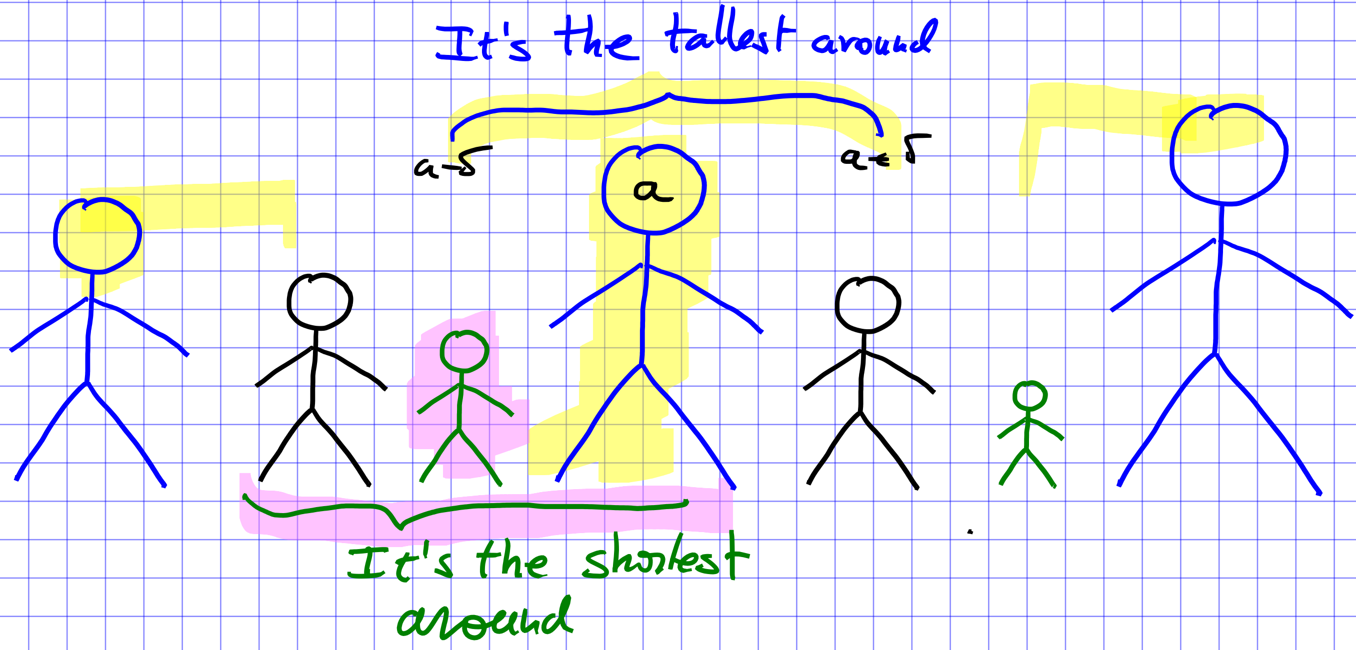
2) In general a function may not attain a maximum or a minimum

3) If it does, it can do it at several points

Theorem

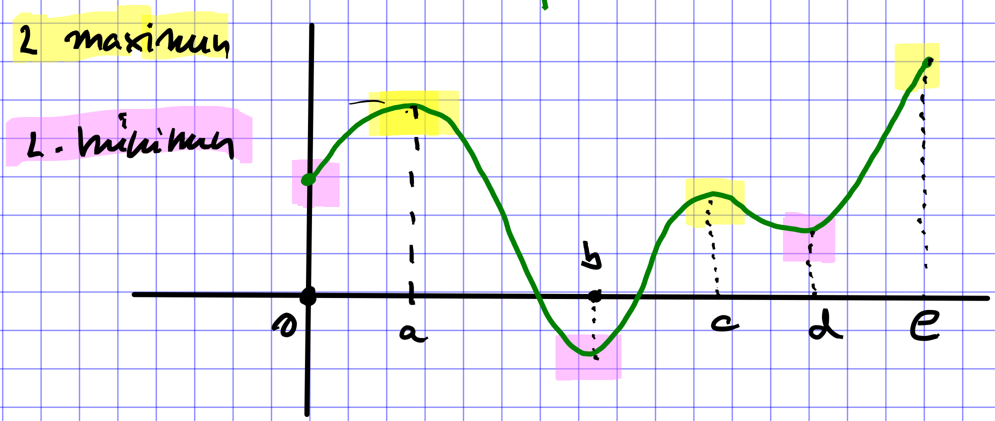
A continuous function on a closed and finite interval $[a,b]$ allways attains a maximum and a minimum on $[a,b]$

Question: How can we find them?

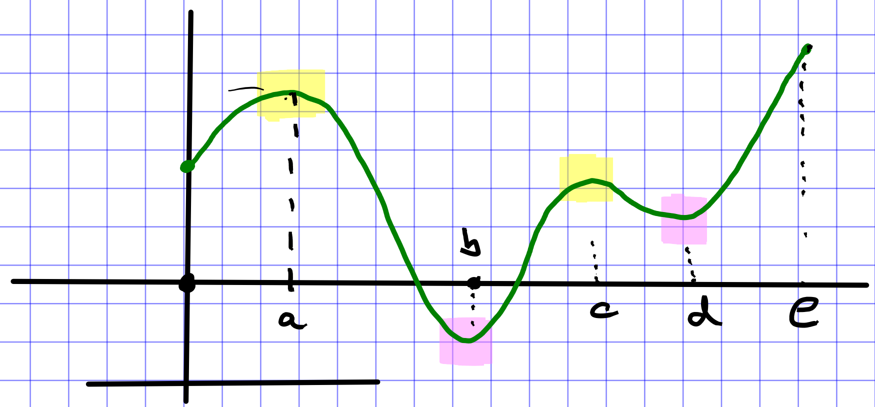


We say that f has a **local maximum at a** if there exists $\delta > 0$ such that $f(x) \leq f(a)$ for all x in $(a-\delta, a+\delta)$

We say that f has a **local minimum at a** if there exists $\delta > 0$ such that $f(x) \geq f(a)$ for all x in $(a-\delta, a+\delta)$



- Note: Maximum/minimum values are also local!!!
- local minimums & maximums are **local extrema**.



Theorem

If $f:[a,b]\to\mathbb{R}$ has a local extremum at x_0 and x_0 is an interior point (i.e. $a < x_0 < b$, $x_0 \in (a,b)$) that is x_0 is not one of the extreme of the interval)

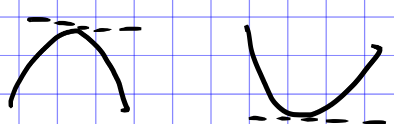
Then x is a critical point. That is the tangent line is horizontal (has slope 0)

$y = f'(x_0)(x - x_0) + f(x_0)$

i.e. $[f'(x_0) = 0 \_ \_ \_]$

Note: 1) The criterion does not include the extremes of the interval

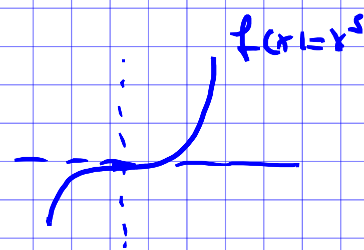
2) If $f'(x_0)=0$, it does not imply that x_0 is a local extremum.



Note: $f(x) = x^3$

$f'(x) = 3x^2 = 0 \iff x = 0$

$x=0$ is a critical point.



Theorem Let f be a continuous function on $[a,b]$.

Suppose that f as a maximum / minimum at $\underbrace{x_0 \in [a,b]}$

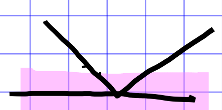
(x_0 belongs to the interval $[a,b]$)

then either

1) x_0 is a critical point
2) x_0 is an endpoint
3) The function f is not differentiable at x_0

So we can follow the following procedure for finding the maximum and/or minimum on $[a,b]$

1) Make a list consisting of
   - Critical points (in (a,b))
   - "Boundary points" (endpoints a and b)
   - Points where f is not differentiable

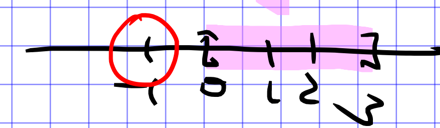


$f(x) = |x|$ is not differentiable at 0

2) Evaluate f at all the points in the list
3) Find the largest and smallest value

Example Find the maximum and minimum value of

$$f(x) = 2x^3 - 3x^2 - 12x + 2 \quad \text{on } [0,3]$$



- Critical points ...

$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 0$$

$$\Leftrightarrow x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$$

$$\Leftrightarrow x = 2 \text{ or } x = -1$$

Note -1 does not lie in the interval $(0,3)$

- Boundary points $x = 0$ and $x = 3$
- We evaluate in all possible points

$f(0) = 2$

$f(3) = 2 \cdot 27 - 27 - 36 + 2 = -7$

$f(2) = 2 \cdot 8 - 3 \cdot 4 - 12 \cdot 2 + 2$

$= 16 - 12 - 24 + 2 = -18$

Answer: The maximum of f on $[0,3]$ is 2 and is attained at 0

The minimum of f on $[0,3]$ is -18 and is attained at $x = 2$

How does f looks like?

Note

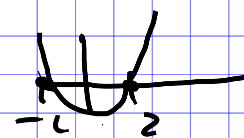
$-$	0	$+$
$\searrow$	$\rightarrow$	$\nearrow$

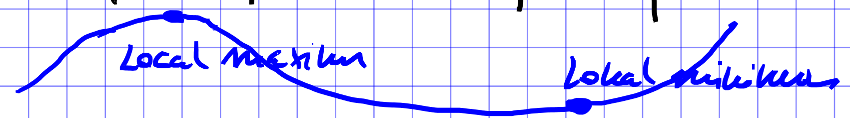
$-$	0	$-$
$\searrow$	$\rightarrow$	$\searrow$

$$f'(x) = 6(x^2-x-2)$$
$$= 6(x+1)(x-2)$$

x	$x<-1$	-1	$-1<x<2$	2	$x>2$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$\rightarrow$	$\searrow$	$\rightarrow$	$\nearrow$

$f'(x)$ looks like

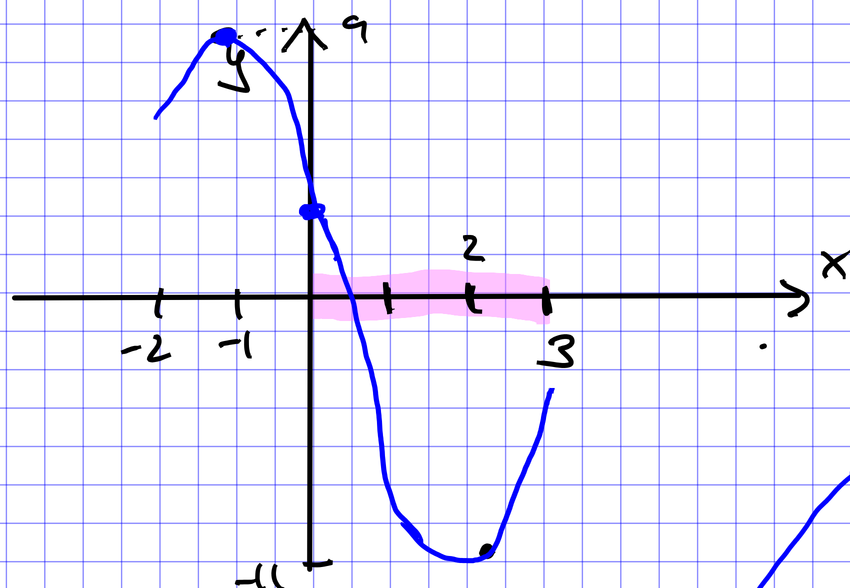




Recall

$f'(x) \geq 0$ on an interval $I \iff f$ is increasing on I

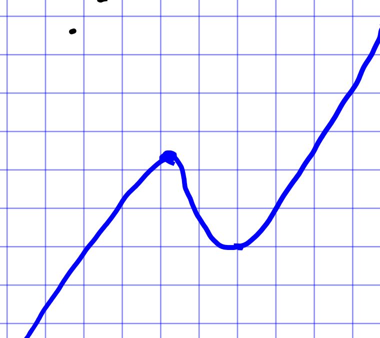
$f'(x) \leq 0$ on an interval $I \iff f$ is decreasing on I



- $f(1) = 2-3-12+2$
  $= -11$

  $f(0) = 2$

  $f(-1) = -2-3+12+2$
  $= 9$



Classification of critical points

If $x \in (a,b)$ $(a<x<b)$ is a critical point of f

1) If the sign of f' changes from positive to negative then x is a local maximum

2) If the sign of f' changes from negative to positive then x is a local minimum

Recall:

$f(x) = ax^2+bx+c$ is convex if $a \geq 0$

$f(x) = ax^2+bx+c$ is concave if $a \leq 0$

We say that f is convex on an interval I if for all $x \in I$ $f''(x) \geq 0$

We say that f is concave on an interval I if for all $x \in I$ $f''(x) \leq 0$

A point x_0 is an inflection point for f if f passes from
convex to concave or concave to convex at x_0

that is

- $(f')'(x)$ passes from positive to negative at x_0

or

- $(f')'(x)$ passes from negative to positive at x_0

that is

f' has a (local) maximum point at x_0

or

f' has a (local) minimum point at x_0

So an inflection point is a point the the slope achives a maximum or a minimum

Example Find all inflection point to

$$f(x) = x^6 - 10x^4$$

Note $f'(x) = 6x^5 - 40x^3$

$$f''(x) = 30x^4 - 120x^2 = 30x^2(x^2-4) = 30x^2(x-2)(x+2)$$

So potential inflection points are $x = 0, 2, -2$

		-2	0	2
$30x^2$				
$x-2$				
$x+2$				
$f''(x)$				
$f(x)$				