Course MM1005

Lecture 8: Integration

Sofia Tirabassi
tirabassi@math.su.se

Salvador Rodriguez Lopez
s.rodriguez-lopez@math.su.se

Lecture Goal and Outcome

Stockholm University

Goals:

- Introduce the notion of integration, properties and methods.

Learning Outcome: At the end of the lecture you will be able to:

- Integrate elementary functions;
- Integrate sum or difference of elementary functions;
- Integrate by substitution and by parts;
- Calculate the signed area under a curve;
- Solve some simple differential equations;
- Calculate improper integrals.

Why you should care

Problem

Let $S(t)$ denote the sales volume of a particular commodity per unit of time, evaluated at time t . In a stable market where no sales promotion is carried out, the decrease in $S(t)$ per unit of time is proportional to $S(t)$. Thus sales decelerate at the constant proportional rate $a > 0$, implying that

$$S'(t) = -aS(t),$$

What is $S(t)$ when sales at time 0 are S_0?

Problem

Let $X = X(t)$ denote the national product, $K = K(t)$ the capital stock, and $L = L(t)$ the number of workers in a country at time t . Under some assumptions one has that

$$K'(t) = 0{,}4e^{0{,}02t}\sqrt{K}, \quad X = \sqrt{KL} \quad L = e^{0{,}04t}$$

Problem: Find the values of K, X, L at time $t = 10$, if we know that $K(0) = 10000$.

Lecture Plan

- §10.1 Indefinite Integrals
- §10.2 Area and Definite Integrals
- §10.3 Properties of Definite Integrals
- §10.5 Integration by Parts
- §10.6 Integration by Substitution
- §10.7 Infinite Intervals of Integration

Section 1
Indefinite Integrals

Indefinite Integrals

Given a function f we want to find all functions F such that $F'(x) = f(x)$ for all x. (F is called a primitive of f)

$f(x)$	$f'(x)$	
x^r	rx^{r-1}	$(r \neq 0)$
e^x	e^x	
$\ln x$	$\frac{1}{x}$	

$f(x)$	$F(x)$
rx^{r-1}	
e^x	
$\frac{1}{x}$	

If F is a primitive of f, all the primitives of f are of the form $F(x) + C$, with $C \in \mathbb{R}$. The set of all primitives of F is denoted by

$$\int f(x)\mathrm{d}x \qquad \text{(indefinite integral)}$$

- $\int e^x \mathrm{d}x = e^x + C$, , where $C \in \mathbb{R}$.
- $\int \frac{1}{x} \mathrm{d}x = \ln x + C$, where $C \in \mathbb{R}$.
- $\int x^s \mathrm{d}x = \frac{1}{s+1} x^{s+1} + C$ for $s \neq -1$, where $C \in \mathbb{R}$.

Given two functions f, g and $a \in \mathbb{R}$

- $\int (f(x) + g(x))\mathrm{d}x = \int f(x)\mathrm{d}x + \int g(x)\mathrm{d}x$
- $\int af(x)\mathrm{d}x = a \int f(x)\mathrm{d}x$

- $\int (3\sqrt{x} + 6x^2 - x^{-1} + e^x)\mathrm{d}x =$

Section 2
Integration methods

Recall two rules we have seen in the course:

- Product-rule for derivatives

$$\frac{\mathrm{d}(u(x)v(x))}{\mathrm{d}x} = \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x) + u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x},$$

-

$$\int \frac{\mathrm{d}w(x)}{\mathrm{d}x}\mathrm{d}x = w(x) + C$$

Combining these two together we have

$$\begin{aligned} u(x)v(x) + C &= \int \frac{\mathrm{d}(u(x)v(x))}{\mathrm{d}x} \\ &= \int \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)\mathrm{d}x + \int u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\mathrm{d}x, \end{aligned}$$

Integration by parts formula

$$\int u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\mathrm{d}x = u(x)v(x) - \int \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)\mathrm{d}x$$

Examples

Integration by parts formula

$$\int u(x)\frac{\mathrm{d}v(x)}{\mathrm{d}x}\mathrm{d}x = u(x)v(x) - \int \frac{\mathrm{d}u(x)}{\mathrm{d}x}v(x)\mathrm{d}x$$

- $\int xe^x \mathrm{d}x$

- $\int \ln x \mathrm{d}x$

Integration by substitution

$$\int f(y(x))y'(x)\mathrm{d}x = \int f(y)\mathrm{d}y$$

Say that $F(y)$ is a primitive for f, and that $y = y(x)$. The chain rule gives

$$\frac{\mathrm{d}F(y(x))}{\mathrm{d}x} = \frac{\mathrm{d}F(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = f(y(x))y'(x)$$

Integrating gives

$$\int f(y(x))y'(x)\mathrm{d}x = F(y(x)) + C = \int f(y)\mathrm{d}y$$

Example 1

Stockholm University

- Find a change of variables such that $f(x)$ is close to the form $g(s(x))s'(x)$.
- Write $\int f(x)\mathrm{d}x$ in the form $\int g(s)\mathrm{d}s = G(s) + C$;
- Write $G(s)$ as $G(s(x))$.

Calculate

1. $$\int \frac{t\ln(t^2+1)}{t^2+1}\mathrm{d}t.$$
2. $$\int \ln(1+2t)(\frac{1}{2}+t)^2\mathrm{d}t$$
3. $$\int (9-x^3)x^2\mathrm{d}x$$
4. $$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\mathrm{d}x$$

Example 2

Recall that we wanted to solve

$$S'(t) = -aS(t) \Leftrightarrow \frac{S'(t)}{S(t)} = -a, \qquad S(0) = S_0,$$

Note that

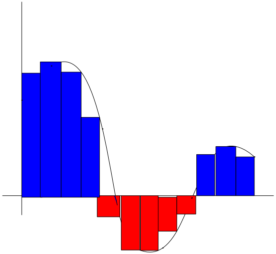
$$\begin{aligned}\frac{S'(t)}{S(t)} = -a \Rightarrow \int(-a)\mathrm{d}t = \int \frac{1}{S(t)}S'(t)\mathrm{d}t = \begin{bmatrix} S = S(t) \\ \mathrm{d}S = S'(t)\mathrm{d}t \end{bmatrix} = \int \frac{1}{S}\mathrm{d}S \\ \Leftrightarrow -at + C = \ln S(t) \Leftrightarrow S(t) = e^{\ln S(t)} = e^{-at+C} = e^C e^{-at} \\ \Leftrightarrow S(t) = Ke^{-at}\end{aligned}$$

So the solution is given by $S(t) = S_0 e^{-at}$.

Section 3
Definite Integral

Definite integral

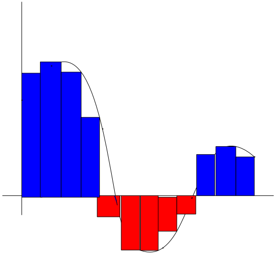
Playing in the casino, we record every hour how much money we gain/loose at the end of that hour. Say we denote the profit (which can be either positive or negative) at time t as $P(t)$. Say that we plot that as below:



What is the net earning after 12 hours?

Definite integral

Playing in the casino, we record every hour how much money we gain/loose at the end of that hour. Say we denote the profit (which can be either positive or negative) at time t as $P(t)$. Say that we plot that as below:

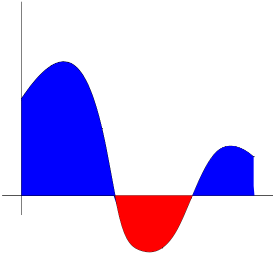


What is the net earning after 12 hours?

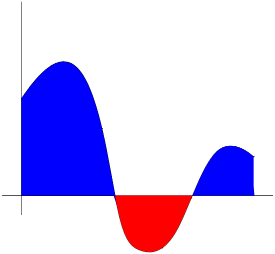
A: The area of the blue rectangles minus the are of the red rectangles.

What about if we record it every second? Or millisecond?

What about if we record it every second? Or millisecond?
A: The area of above the x-axis and under the graph minus area under the x-axis and above the graph.



What about if we record it every second? Or millisecond?
A: The area of above the x-axis and under the graph minus area under the x-axis and above the graph.



Let $f : [a, b] \to \mathbb{R}$. We define the signed area under the graph of f the difference between the area over the x-axis, minus that under the x-axis. We denote that area as

$$\int_a^b f(x)\mathrm{d}x := A_+ - A_-.$$

How can we calculate $\int_a^b f(s)\mathrm{d}s$

Stockholm University

Fundamental Theorem of Calculus

Let $f : [a, b] \to \mathbb{R}$ be continuous. Define, for all $x \in [a, b]$ the area integral $A(x) := \displaystyle\int_a^x f(s)\mathrm{d}s$,

1. $A(x)$ is a primitive for f, i.e. $A'(x) = f(x)$.
2. If $F(x)$ is any primitive for f

$$\int_a^b f(s)\mathrm{d}s = F(b) - F(a) = [F(x)]_a^b,$$

Example: We can use it to calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{x^2} e^s \mathrm{d}s\right) =$$

Calculate the following integrals

1. (2019) $\int_{6}^{49} \frac{1}{2x-5} \mathrm{d}x$
2. (2018) $\int_{-1}^{1} \frac{x^2}{2} \mathrm{d}x$
3. (2017) $\int_{1}^{e^2} x^3 \ln x \mathrm{d}x$

Suppose that when it is t years old, a particular machine generates revenue at the rate $R'(t) = 5000 - 20t^2$ \$/year and that operating and servicing costs related to the machine accumulate at the rate $C'(t) = 2000 + 10t^2$ \$/year.

1. How many years pass before the profitability of the machine begins to decline?
2. Compute the net earnings generated by the machine over the time period determined in the first part.

Section 4
Improper Integrals

Improper integral

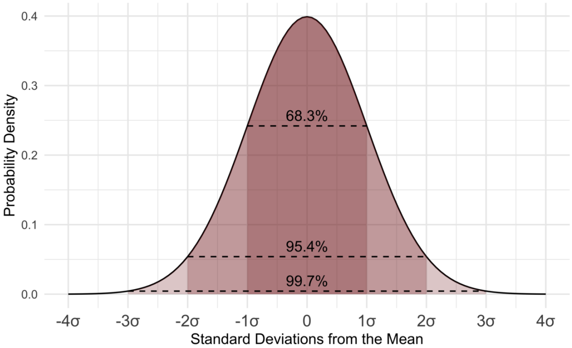
1. Let f be a continuous function on $[a, \infty)$. Define
$$\int_a^\infty f(x)\,dx \quad \text{to be} \quad \lim_{b\to\infty}\int_a^b f(x)\,dx,$$
2. Let f be a continuous function on $(-\infty, b]$. Define
$$\int_{-\infty}^b f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^b f(x)\,dx,$$
3. Let f be a continuous function on $(-\infty, \infty)$. Let c be any real number; define
$$\int_{-\infty}^\infty f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx,$$

An improper integral is said to **converge** if its corresponding limit exists; otherwise, it **diverges**.

Motivational example

If $X \sim N(\mu, \sigma)$ then its distribution function is given by

$$P\,(X \leq t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x,$$



Examples

1. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$
2. $\displaystyle\int_1^{\infty} \frac{1}{x}\,\mathrm{d}x$
3. $\displaystyle\int_0^{+\infty} e^{-x}\,\mathrm{d}x$
4. $\displaystyle\int_0^{1} \frac{1}{\sqrt{x}}\,\mathrm{d}x$

Thank you for your attention!