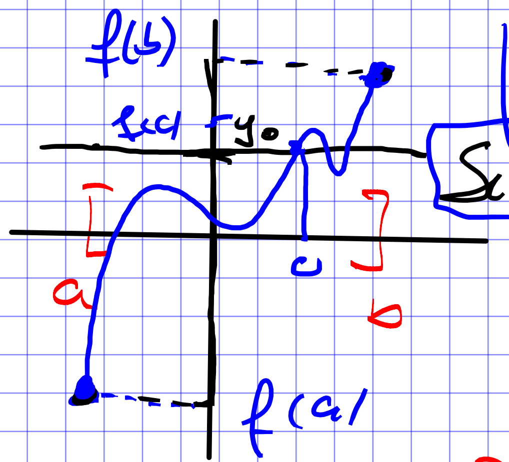
Intuitively: A function f is continuous on $[a,b]$ if we can draw its graph without lifting the pen from the paper.

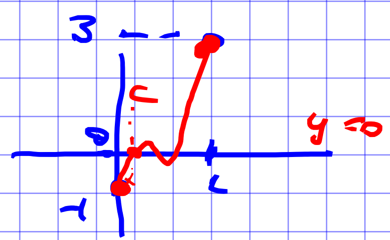
Intermediate Value theorem

Let $f:[a,b]\to\mathbb{R}$ be continuous.
For every value y_0 in between $f(a)$ and $f(b)$ there exists $c\in[a,b]$ such that $f(c)=y_0$.



Let $f(x)=x^6+3x^2-2x+1$

Since $\boxed{f(0)=-1}$, $\boxed{f(1)=3}$, there exists $c\in(0,1)$ such that $\boxed{f(c)=0}$



Q: How we can find c?
E.g. Newton's method

Classification of critical points (i.e. $f'(x_0)=0$)

	$x<x_0$	x_0	$x>x_0$
f'	+	0	−
f	↗	→	↘

If $x \in (a,b)$ ($a<x<b$) is a critical point of f

1) If the sign of f' changes from positive to negative then x is a local maximum

2) If the sign of f' changes from negative to positive then x is a local minimum

Example

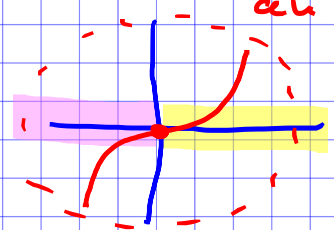
$f'(x) \geq 0$ on I $\iff$ $f \nearrow$ on I

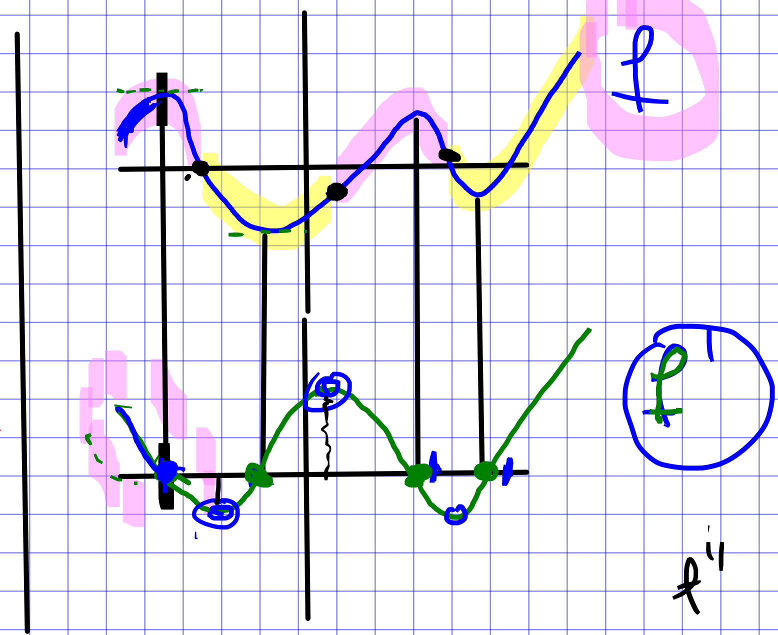
$f'(x) \leq 0$ on I $\iff$ $f \searrow$ on I

$f(x) = x^3$

$f'(x) = 3x^2$ ($x=0$ is a critical point) but not an extremum

x	$x<0$	0	$x>0$
f'	+	0	+
f	↗	→	↗





Recall: $f(x) = ax^2+bx+c$ is convex if $\boxed{a \geq 0}$ ∪ $\iff f''(x) \geq 0$

$f(x) = ax^2+bx+c$ is concave if $\boxed{a \leq 0}$ ∩ $\iff f''(x) \leq 0$

$f'(x) = 2ax+b$

$[f''(x) = 2a]$

- - -

We say that f is convex on an interval I if for all $x \in I$ $f''(x) \geq 0$

We say that f is concave on an interval I if for all $x \in I$ $f''(x) \leq 0$

A point x_0 is an inflection point for f if f passes from convex to concave or concave to convex at x_0

that is

- $(f')'(x)$ passes from positive to negative at x_0

or

- $(f')'(x)$ passes from negative to positive at x_0

that is

f' has a (local) maximum point at x_0

or

f' has a (local) minimum point at x_0

So an inflection point is a point the the slope achives a maximum or a minimum

Example Find all inflection point to

$$f(x) = x^6 - 10x^4$$

Note $f'(x) = 6x^5 - 40x^3$

$$f''(x) = 30x^4 - 120x^2 = 30x^2(x^2-4) = 30x^2(x-2)(x+2)$$

So potential inflection points are $x = 0, 2, -2$

		-2		0		2	
$30x^2$							
$x-2$							
$x+2$							
$f''(x)$							
$f(x)$							

Test yourself! Attempt to solve the following exercises

Consider the function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$.

(a) Find the critical points;

(b) Find the intervals on which f is increasing and decreasing;

(c) Is there any inflection point? Argument your answer.

(d) Give the maximum and minimum value of f on the interval $[0, 3]$.

2015

5. Let $g(x) = \sqrt[3]{1+x}$.

(a) Find $T_2(x)$, the 2nd-order Taylor approximation of g around $x = 0$.

(b) Show that the error term $R_3(x) = g(x) - T_2(x)$ satisfies $|R_3(x)| \leq \frac{5x^3}{81}$, for $x \geq 0$.

(c) Use $T_2(x)$ to compute $\sqrt[3]{(1002)}$ to 7 decimal places. *Hint: use* $1002 = 1000(1 + \frac{2}{1000})$.

Find the following limits, if they exist.

(a) $\displaystyle\lim_{x\to\infty} \frac{x^2}{2-x} + x$

(b) $\displaystyle\lim_{x\to 0} \frac{e^{x^2} - 1}{x \ln(2+x)}$

Consider the curve given implicitly by the equation $y^3 + x^2 y = 10$. (This defines y as a function of x.)

(a) Find $y(1)$.

(b) Find the equation of the tangent line to this curve at the point $(1, y(1))$.

2018

Lecture 7

$$\left\{\begin{array}{lcl} f(x) & \longrightarrow & f'(x) \\ x^r & & r\,x^{r-1} \\ e^x & & e^x \\ \ln x & & \frac{1}{x} \end{array}\right.$$

We would like to look at the following question

Given $f(x)$ find $F(x)$ such that $F'(x) = f(x)$

(F is called a primitive for f)

<u>Example</u>

Given $f(x) = x^3$. Can we find $F(x)$ such that $F'(x) = x^3$?

Note that if we take $F(x) = A x^4$ we have that

$$F'(x) = 4A\,x^3 \stackrel{?}{=} x^3 = 1 \cdot x^3$$

so it's enough to take $\left[A = \frac{1}{4}\right]$

$\left[F(x) = \frac{1}{4}x^4 \text{ } \underline{\text{satisfies that}} \text{ } \underline{F'(x) = x^3}\right]$

Observe also that the same is satisfied by

$$F(x) = \frac{1}{4}x^4 + 3 \quad \text{or} \quad F(x) = \frac{1}{4}x^4 - 10$$

indeed

$$F(x) = \frac{1}{4}x^4 + C \quad \text{where } C \in \mathbb{R}.$$

F(x) is a primitive for f(x)

- F(x) is a primitive for f(x) if for all x $F'(x) = f(x)$
- If a function f has a primitive F(x) all other primitives are of the form F(x) + C

The collection of all primitives of f is denoted by

$\int$ aga

"s"

$$\int f(x)\,dx = \underline{\underline{\text{indefinite}}} \text{ integral of } f$$

Example

$$\int x^3\,dx = \frac{x^4}{4} + C \qquad C \in \mathbb{R}.$$

Properties

i) $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

ii) $\int A f(x)\,dx = A \int f(x)\,dx$

$r \neq -1$ $\quad \int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1) \quad C \in \mathbb{R}$

$$\int x^{-1}\,dx = \ln|x| + C \qquad C \in \mathbb{R}$$

$$\int e^x\,dx = e^x + C \qquad C \in \mathbb{R}.$$

Example

i) $\int (\sqrt{x} + 3x^2)\,dx = \int x^{\frac{1}{2}}\,dx + 3\int x^2\,dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_1 + 3\frac{x^3}{3} + 3C_2 = \frac{x^{3/2}}{3/2} + x^3 + C \qquad C \in \mathbb{R}.$

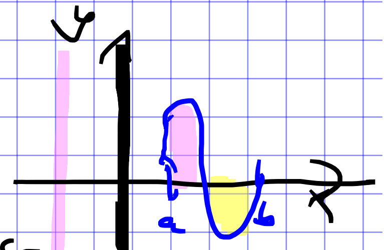
$= \frac{2}{3} x^{3/2} + x^3 + C \qquad C \in \mathbb{R}.$

ii) $\int 2^x\,dx$

$2^x = e^{\ln(2^x)} = e^{x(\ln 2)} \quad \Big| \quad \int 2^x\,dx = \int e^{(\ln 2)x}\,dx$

Calculus of Areas

Given a function $f:[a,b]\to\mathbb{R}$ we consider the region between the graph of f and the x-axis limited by a and b



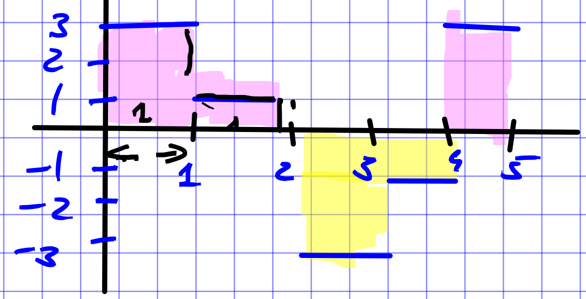
Sum

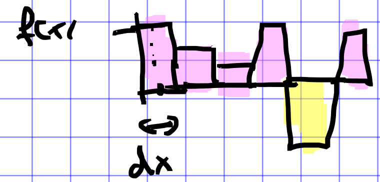
$|f(x)|$ $\int_a^b f(x)\,dx$

Example

We play at the casino and we plot every hour how much we gain / loose

$3\cdot 1+1\cdot 1+(-3)+(-1)+3 = 3$

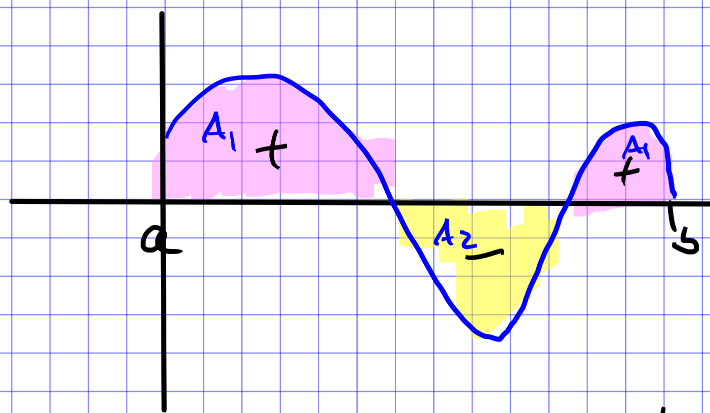




the area over the x-axis represents positive profit & under negative profit.

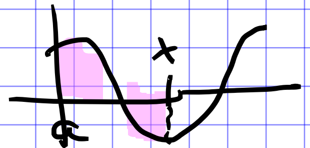
[The total "area" represents the profit (if positive) or loses (if negative)]

In general ; we define the signed area as the difference between the area over the x-axis minus that under the axis



We denote that area

$$\boxed{\int_a^b f(x)\,dx} = A_1 - A_2$$



Theorem (Fundamental theorem of Calculus)

Let $f : [a,b] \to \mathbb{R}$ be continuous

i) For every $x \in [a,b]$ the area integral

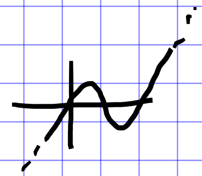
$\boxed{A'(x) = f(x)}$ $\qquad \boxed{A(x) = \int_a^x f(s)\,ds}$ (Continuous functions have a <u>primitive</u>)

is a primitive of f

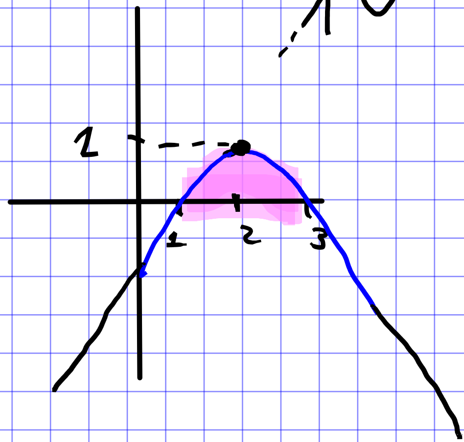
ii) If F is (another) primitive of f

$$\boxed{\int_a^b f(s)\,ds = F(b) - F(a) =: F(x)\Big|_a^b}$$

<u>Example</u> Find the area over the x-axis and under the graph



$$\left[y = -x^2 + 4x - 3 \right]$$



We first need to visualise that area

$y(x)$ describes a concave parabola

$$(y''(x) = -2 < 0)$$

that has a global maximum at $x=2$

$\lim_{x\to+\infty} y(x) = -\infty$ & $\lim_{x\to-\infty} y(x) = -\infty$

$$\left(\underline{y'(x)} = \underline{-2x+4} = 0 \iff x = 2 \right. \text{ (critical point)} \Big)$$

$$y(2) = -4+8-3 = 1$$

x	$x<2$	2	$x>2$
y'	$+$	0	$-$
y	$\nearrow$	$-$	$\searrow$

$x=2$ local maximum.

and <u>intersects the x-axis</u> at $x = 1$ and $x = 3$

$$y(x) = 0 \iff x = \frac{-4 \pm \sqrt{16-12}}{-2} = \frac{-4 \pm \sqrt{4}}{-2} = \frac{-4 \pm 2}{-2} = \begin{cases} \frac{-2}{-2} = 1 \\ \frac{-6}{-2} = 3 \end{cases}$$

We want to calculate

$$\int_1^3 (-x^2+4x-3)\,dx = A$$

, so we need <u>a</u> primitive for $f(x) = -x^2+4x-3$

$$\int (-x^2+4x-3)\,dx = -\int x^2\,dx + 4\int x^1\,dx - 3\int \overset{x^0}{1}\,dx$$

$$= -\frac{x^3}{3} + 4\frac{x^2}{2} - 3\frac{x^1}{1} + C$$

$$= -\frac{x^3}{3} + 2x^2 - 3x + C \qquad C \in \mathbb{R}.$$

So $\left[F(x) = -\frac{x^3}{3} + 2x^2 - 3x\right]$ is <u>a primitive for $f(x)$</u>

By the <u>Fundamental theorem of calculus</u>

$$\int_1^3 (-x^2+4x-3)\,dx = F(3) - F(1) \left(=: F(x)\Big|_1^3 \right)$$

$$\left[F(3) = -\frac{3^3}{3} + 2\cdot 3^2 - 3^2 = -3^2 + 2\cdot 3^2 - 3^2 = 3^2(-1+2-1) = 0\right]$$

$$\left[F(1) = -\frac{1}{3} + 2 - 3 = -\frac{1}{3} - 1 = -\frac{4}{3}.\right]$$

Then

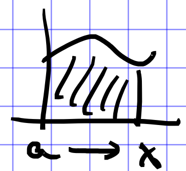
$$\int_1^3 (-x^2+4x-3)\,dx = 0 - \left(-\frac{4}{3}\right) = \frac{4}{3}.$$

Properties of the definite integral

i) $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

ii) $\int_a^b A f(x)\,dx = A \int_a^b f(x)\,dx$

iii) $\int_a^a f(x)\,dx = 0$



iv) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ $\qquad \left[\int_1^2 f(x)\,dx = -\int_2^1 f(x)\,dx\right]$

(v) $\left(\int_a^x f(s)\,ds\right)' = f(x)$

- - - - - -

Example Calculate

$$\frac{dF}{du} = e^u$$

$$\frac{d}{dx}\left[\underbrace{\int_0^{x^2} e^s\,ds}\right] = \left[\overbrace{F(u) = \int_0^u e^s\,ds}, \; u(x) = x^2\right] = \frac{d}{dx}\left[\underbrace{F(x^2)}\right]$$

$$= \left(\frac{d}{du}F\right)(\underbrace{x^2}_{u(x)}) \cdot \underbrace{2x}_{\frac{du}{dx}} = e^{x^2} \cdot 2x$$

$$\int_0^{x^2} e^s\,ds = G(x^2) - G(0) = e^{x^2} - e^0 = e^{x^2} - 1$$

where $G(u)$ is a primitive of e^u $\left| \; \frac{d}{du}(e^u) = e^u \rightarrow \int e^u\,du = e^u + C \right]$

$$\frac{d}{dx}\left(e^{x^2} - 1\right) = e^{x^2} \cdot 2x$$

Integration by parts

Note that $\boxed{\int f'(x)\,dx = f(x) + C}$, $C \in \mathbb{R}$.

We have

$$(u\cdot v)'(x) = u'(x)\,v(x) + u(x)\,v'(x)$$

$\int (u\cdot v)'\,dx = u\cdot v + C$

Then

$$\left[(u\cdot v)(x) = \int u'(x)\,v(x)\,dx + \int u(x)\,v'(x)\,dx\right]$$

That is

$$\int u(x)\,v'(x)\,dx = (u\cdot v)(x) - \int u'(x)\,v(x)\,dx$$

[Integration by parts formula]

Example

i) $\int x e^x \, dx = \left[\begin{matrix} u = x & u' = 1 \\ v' = e^x & v = e^x \end{matrix}\right] = x e^x - \int 1 \cdot e^x \, dx$

$= x e^x - e^x + C \quad C \in \mathbb{R}.$

ii) $\int \ln x \, dx = \int 1 \cdot \ln x \, dx = \left[\begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = 1 & v = x \end{array} \right]$

Integration by substitution

$$\int f(u(x))\, u'(x)\, dx = \int f(u)\, du$$

If $F(u)$ is a primitive for $f(u)$ and $u = u(x)$

$$\frac{d}{dx}\left[F(u(x)) \right] = F'(u(x)) \cdot u'(x) = f(u(x)) \cdot u'(x)$$

So

$$F(u(x)) + C = \int f(u(x))\, u'(x)\, dx$$

Example

i) $$\int (x^2+1)^{2021} x \; dx = \left[\begin{array}{l} u(x) = x^2+1 \\ u'(x) = 2x \end{array} \right]$$

$$= \int (u(x))^{2021} \frac{1}{2} u'(x)\, dx = \frac{1}{2} \int u^{2021}\, du = \frac{u^{2022}}{2 \cdot (2022)} + C$$

$$= \frac{(x^2+1)^{2022}}{4044} + C$$

Integration by substitution (Procedure)

- ▷ Find a change of variable $u(x)$ so that $f(x)$ is close to the form $g(u(x)) \cdot u'(x)$
- ▷ Write $\int f(x)\,dx$ in the form $\int g(u)\,du$
- ▷ Integrate $\int g(u)\,du = G(u) + C$
- ▷ Write back $G(u)$ as $G(u(x))$

ii) $\int x^3 (x^2+1)^{1/2}\,dx = \frac{1}{2}\int x^2 \,(x^2+1)^{1/2}\, 2x\,dx = \left[\begin{array}{l} u(x) = x^2+1 \rightarrow x^2 = u-1 \\ u'(x) = 2x \end{array}\right]$

iii) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Integrals of rational functions $\int \frac{P(x)}{Q(x)}\,dx$

1st step : If needed, use polynomial division to write

$$\frac{P(x)}{Q(x)} = r(x) + \frac{q(x)}{Q(x)} \quad \text{with } \operatorname{degree}(q) < \operatorname{degree}(Q)$$

So

$$\int \frac{P(x)}{Q(x)}\,dx = \int r(x)\,dx + \int \frac{q(x)}{Q(x)}\,dx$$

2nd step To integrate $\int \frac{q(x)}{Q(x)}\,dx$ one needs to rewrite $\frac{q(x)}{Q(x)}$ by possibly factoring $Q(x)$ (Partial fractions)

Example Calculate $\int \frac{x^4+3x-4}{x^2+2x}\,dx$

1) Polynomial division gives

$$\int \frac{x^4+3x-4}{x^2+2x}\,dx = \underbrace{\int (x^2-2x-7)\,dx}_{\frac{x^3}{3}-x^2-7x+C_1} + \int \frac{14x+4}{x^2+2x}\,dx$$

Now $x^2+2x = x(x+2)$ so one can re-write for suitable $A, B \in \mathbb{R}$

$$\frac{14x+4}{x^2+2x} = \frac{14x+4}{x(x+2)} \overset{?}{=} \frac{A}{x} + \frac{B}{x+2} = \frac{A(x+2)+Bx}{x(x+2)} = \frac{(A+B)x+2A}{x(x+2)}$$

We look for A, B satisfying

$$\left.\begin{array}{l} A+B = 14 \\ 2A = 4 \end{array}\right\} \Longleftrightarrow \boxed{\begin{array}{l} B = 12 \\ A = 2 \end{array}}$$

In this way

$$\int \frac{14x+4}{x^2+2x}\,dx = A \underbrace{\int \frac{dx}{x}}_{\ln|x|+C'} + B \underbrace{\int \frac{dx}{x+2}}_{\ln|x+2|+C'}$$

$$= \ln x^2 + \ln(x+2)^{12} + C_1 = \ln\left(x^2(x+2)^{12}\right) + C'$$

Answer:

$$\int \frac{x^4+3x-4}{x^2+2x}\,dx = \frac{x^3}{3} - x^2 - 7x + \ln\left(x^2(x+2)^{12}\right) + C'$$

Improper integrals

Sometimes makes sense considering integrals of the form

$$\int_a^{+\infty} f(x)\,dx \quad \text{or} \quad \int_{-\infty}^{+\infty} f(x)\,dx \quad \text{or} \quad \int_{-\infty}^{a} f(x)\,dx$$

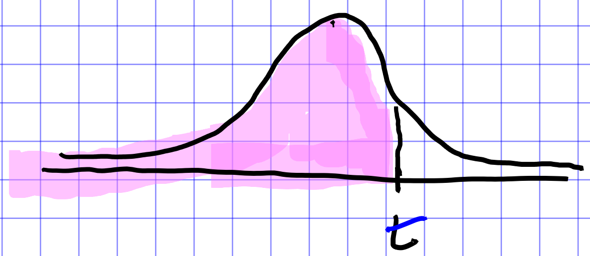
for f continuous

Example:

If $g(x) = \mathbb{1}_A(x)$

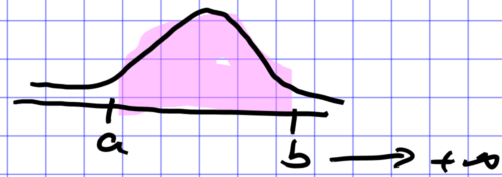
$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x)\, \underbrace{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}_{\text{Normal distribution of mean } \mu \text{ and variance } \sigma} dx = \mathbb{P}(X \in A) \qquad [X \sim N(\mu, \sigma^2)]$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \mathbb{P}(X \leq t) \quad \text{(Distribution function)}$$



One defines

$$\int_a^{+\infty} f(x)\,dx := \lim_{b\to+\infty} \int_a^b f(x)\,dx$$



(provided the limit exists)

Analogously we define

$$\int_{-\infty}^{b} f(x)\,dx := \lim_{a\to-\infty} \int_a^b f(x)\,dx \quad \& \quad \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{+\infty} f(x)\,dx$$

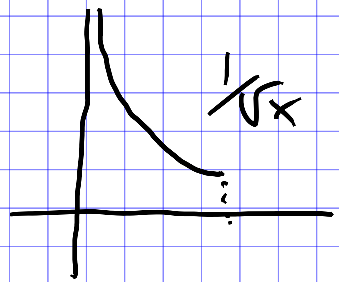
Example $\lambda > 0$

$$\int_0^{+\infty} \lambda e^{-\lambda x}\,dx = \lim_{b\to+\infty} \int_0^b \lambda e^{-\lambda x}\,dx = \lim_{b\to+\infty} -e^{-\lambda x}\Big|_0^b$$

$$= \lim_{b\to+\infty} (1 - e^{-\lambda b}) = 1.$$

Integrals of unbounded functions

Example The function $\frac{1}{\sqrt{x}}$ is unbounded on $[0,1]$



but we would like to give sense to

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^+} 2\sqrt{x}\,\Big|_a^1 =$$

$$= \lim_{a\to 0^+} (2-2\sqrt{a}) = 2.$$

If $\lim_{x\to a^+} f(x) = \pm\infty$ one defines

$$\int_a^b f(x)\,dx = \lim_{x\to a^+} \int_x^b f(x)\,dx$$

assuming that the limit exists.

A *geometric progression* is a sequence of numbers (finite or infinite)

$$a_1, a_2, a_3, \ldots$$

with the property that the *ratio* $\frac{a_{n+1}}{a_n}$ is the same for all n.
In other words, a geometric progression is a sequence of the following form

$$a, ar, ar^2, \ldots, ar^{n-1}, \ldots$$

a is the *initial term* of the sequence and r is the *ratio*.

Example

i) $3, -6, 12, -24, \ldots$ is a geometric progression of ratio (-2) and initial term 3

ii) $\underbrace{k}_{\text{Initial Capital}} \underbrace{\left(1 \pm \frac{r}{100}\right)^n}_{\text{Compound interest after } n\text{-years}}$ $n = 0, 1, 2, \ldots$ is a geometric progression

$1 \pm \frac{r}{100}$ = growth factor

Geometric Series

This year a firm has an annual revenue of \$100 million that it expects to increase by 16% per year throughout the next decade. How large is its expected revenue in the tenth year, and what is the total revenue expected over the whole period?

→ The expected revenue in the tenth year

$$100\left(1+\frac{16}{100}\right)^9 =: k\, r^9$$

$k = 100$

$r = 1+\dfrac{16}{100} = 1.16$

→ The total revenue over the decade

(I'm lazy to do the sum term by term)

$$R = k + k\cdot r + k\cdot r^2 + k r^3 + \dots + k r^9$$

$$= k\,(\underbrace{1+r+r^2+r^3+\dots+r^9}_{\text{lets call this } S(10)})$$

Note that

$$r\,S(9) = r+r^2+\dots+r^9 + r^{10} = S(9) - 1 + r^{10}$$

So solving for $S(9)$ we get

$$(r-1)\,S(9) = r^{10}-1 \iff S(9) = \frac{r^{10}-1}{r-1} \quad (r \neq 1)$$

So $$R = k\,\frac{r^{10}-1}{r-1} = 100 \cdot \frac{\left(1+\frac{16}{100}\right)^{10}-1}{16/100} = 100^2\left((1.16)^{10}-1\right)$$

In general

$$k + kr + \dots + kr^n = k\,\frac{r^{n+1}-1}{r-1}$$

Q: How many years will take for the revenue to be at least $10\cdot k$ millions?

One needs to find the smallest n such that

$$k\,\frac{r^{n+1}-1}{r-1} \geq 10\cdot k$$

$$\Leftrightarrow\ r^{n+1}-1 \geq 10(r-1) \Leftrightarrow r^{n+1} \geq 10(r-1)+1$$

$$\Leftrightarrow\ (n+1)\ln r \geq \ln\left(10(r-1)+1\right)$$

$$\Leftrightarrow n+1 \geq \frac{\ln(10(r-1)+1)}{\ln r}$$

In the case above $r = 1 + \frac{16}{100} = 1{,}16$

$$n+1 \geq \frac{\ln(2{,}6)}{\ln(1{,}16)} \simeq 6.4378$$

So $n+1$ has to be at least 7.

(Answer: We need to wait 7 years for the revenues to be larger than 10.k