Course Name

Lecture 10: Functions of several variables

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Questions?

Lecture Goal and Outcome

Goals:

- Identify functions of two variables
- Determine the first-order partial derivatives of functions
- Determine second-order partial derivatives,
- Use the chain rule
- Find stationary points for a function.

Learning Outcome: After today you will be able to solve problems like the following:

Problem

Find all the stationary points of the function
$f(x, y) = (4 - x - y^2)(x + 1)$.

Why you should care

Problem

A firm produces two different kinds of commodities A, B. The daily cost of producing x units of A and y units of B is given by

$$C(x, y) = 4x^2 + xy + y^2 + 400x + 200y + 500,$$

If the firms sells its production at a price per unit 15 for A and 9 for B, how can we find the daily production levels x, y that maximise the profit?
If x and y are related, say $y = y(t)$ and $x = x(t)$, how can we maximise that profit?

Cobb-Douglas model

Productivity depends on several variables such as labor, capital land or raw materials. For instance, the Cobb-Douglas model depending only capital and Labour is given by

$$Y = f(K, L) := AK^{\alpha}L^{\beta},$$

where

- $Y \equiv$ total production
- $L \equiv$ Labour input
- $K \equiv$ Capital invested
- $A \equiv$ total factor productivity.
- α, β are the output elasticities.

Lecture Plan

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- §14.1 Functions of Two Variables
- §14.2 Partial Derivatives with Two Variables
- §14.3 Geometric Representation
- §14.4 Surfaces and Distance
- §14.5 Functions of More Variables
- §15.1 A Simple Chain Rule
- §17.1 Two Variables: Necessary Conditions
- (§17.7 Comparative statistics and the envelope Theorem)

Section 1
Functions of several variables

Functions of two variables

A company produces two goods (A, B) in quantities x, y respectively. The total cost of producing them is given by

$$C(x, y) = 1000 + 20x + 40y,$$

Let $D \subset \mathbb{R}^2$. A function $f : D \to \mathbb{R}$ is a rule, that for each $(x, y) \in D$ it gives us a value $f(x, y)$. D is called the domain of definition of f.

Examples:

1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = x^2 - y^2$
3. $f(x, y) = x^2 e^{x+y}$
4. $f(x, y) = 1/(x^2 + 2y^2 + 1)$

The domain of definition of these functions is ...
If $D \subset \mathbb{R}^n$, we say that f is a function of n-variables.

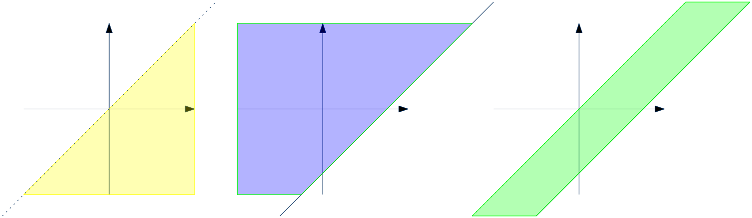
Domain of definition

Let $f(x,y) = \ln(x-y) + \sqrt{1-x+y}$.
The domain of definition of

- $\ln(x-y)$ is all points (x,y) such that $x-y>0$ (i.e. $y<x$).
- $\sqrt{1-x+y}$ is all points (x,y) such that $1-x+y \geq 0$ (i.e. $y \geq x-1$).

Hence, the domain of definition of f is the set of all points (x,y) satisfying both conditions:

$$\{(x,y) : x-1 \leq y < x\},$$



Partial derivatives

We define the partial derivative of f with respect to x at (x_0, y_0) as

$$\frac{\partial f}{\partial x}(x_0, y_0) = f'_x(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

> We fix y_0, and differentiate the function as a one-variable function in x

Similarly we define the partial derivative of f with respect to y at (x_0, y_0) as

$$\frac{\partial f}{\partial y}(x_0, y_0) = f'_y(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

Example: If $f(x, y) = x^2 + xy + 3y^2$ then

$$f'_x(x, y) = 2x + y, \qquad f'_y(x, y) = x + 6y$$

Examples

We can use all the rules we have learnt to compute partial derivatives:
If $f(x,y) = (x^2 + y)e^{xy}$ then

$$\begin{aligned} f'_x(x,y) &= \left[\frac{\partial}{\partial x}(x^2+y)\right] e^{xy} + (x^2+y)\left[\frac{\partial}{\partial x} e^{xy}\right] \\ &= [2x]\, e^{xy} + (x^2+y)\,[ye^{xy}] \\ &= (2x + x^2 y + y^2)e^{xy} \end{aligned}$$

Similarly

$$f'_y(x,y) =$$

Higher order derivatives

We can define

$$f''_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) \qquad f''_{yx}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)$$

$$f''_{xy}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) \qquad f''_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y)$$

Example If $f(x,y) = x^3 e^{y^2}$ then $f'_x(x,y) = 3x^2 e^{y^2}$, $f'_y(x,y) = 2yx^2 e^{y^2}$,

$$f''_{xx}(x,y) = 6x^2 e^{y^2} \qquad f''_{yx}(x,y) = 6x^2 y e^{y^2}$$

$$f''_{xy}(x,y) = 6x^2 y e^{y^2} \qquad f''_{yy}(x,y) =$$

If the second partial derivatives exists and are continuous then
$$f''_{yx}(x,y) = f''_{xy}(x,y)$$

Geometric representation

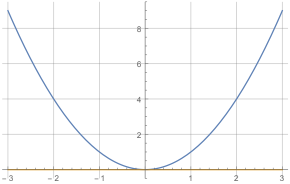
A real number x — A point on the line $\mathbb{R}$

A pair of numbers (x, y) — A point on the plane $\mathbb{R}^2$

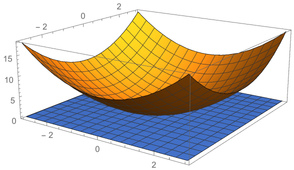
A triple of number (x, y, z) — A point in space $\mathbb{R}^3$

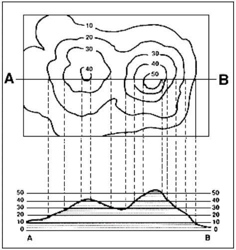
The graph of $f : D \to \mathbb{R}$ is the set of points

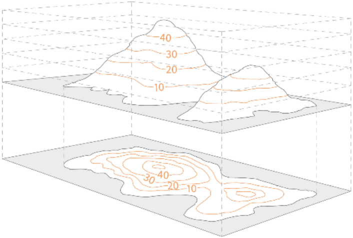
If $D \subset \mathbb{R}$
in $\mathbb{R}^2$ of the form $(x, f(x))$
with $x \in D$.

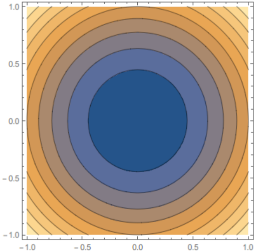


If $D \subset \mathbb{R}^2$
in $\mathbb{R}^3$ of the form $(x, y, f(x, y))$ with $(x, y) \in D$.









A way of representing graphically a function f is by plotting in the (x, y)-plane the sets of points satisfying that

$$f(x, y) = c$$

for a range of constants c. These are called level curves of f.

Distance

Given two points in the plane $X = (x, y)$, $Y = (a, b)$ the distance between them is

$$d(X, Y) := \sqrt{|x - a|^2 + |y - b|^2},$$

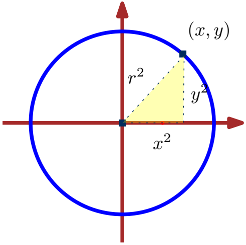
So, if $Y = (0, 0)$, $f(x, y) := \sqrt{x^2 + y^2}$ represents the distance of (x, y) to the origin. The level curves of the form

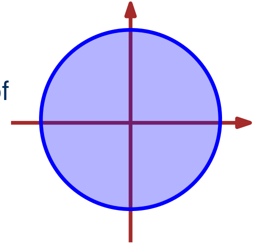
$$x^2 + y^2 = r^2, \quad r > 0$$

are circle of radius c. Also, the sets of points (x, y) such that

$$x^2 + y^2 \leq r^2, \quad r > 0$$

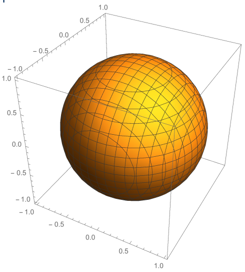
are disks of radius r.





Given two points in the plane $X = (x, y, z)$, $Y = (a, b, c)$ the distance between them is

$$d(X, Y) := \sqrt{|x-a|^2 + |y-b|^2 + |z-c|^2}$$



So, $\sqrt{x^2 + y^2 + z^2}$ represents the distance of (x, y, z) to the origin.
The set of points (x, y, z) such that

$$x^2 + y^2 + z^2 = r^2, \quad r > 0$$

is a sphere of radius r. Also, the set of points (x, y, z) such that

$$x^2 + y^2 + z^2 \leq r^2, \quad r > 0$$

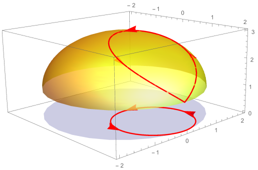
is a (solid) ball of radius r.

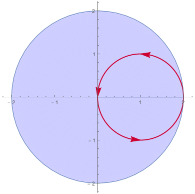
Section 2
The chain rule

Chain rule

Assume that $f(x, y)$ is a function of two variables and both $x = x(t)$, and $y = y(t)$ are functions of one variable t. Then we can consider the one-variable function

$$g(t) := f(x(t), y(t))$$





The points $t \mapsto (x(t), y(t))$ can be thought describing a curve on the domain of f. So g describes the values of f along that curve.

The chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = \frac{\partial}{\partial x} f(x(t), y(t)) \frac{\mathrm{d}}{\mathrm{d}t} x(t) + \frac{\partial}{\partial y} f(x(t), y(t)) \frac{\mathrm{d}}{\mathrm{d}t} y(t)$$

Example

If $f(x, y) = x^2 y + 3y$, and $\begin{cases} x(t) = t^2 \\ y(t) = t^3 \end{cases}$ $t \in \mathbb{R}$ then by the chain rule

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f^{(} x(t), y(t)) &= \frac{\partial}{\partial x} f(x(t), y(t)) \frac{\mathrm{d}}{\mathrm{d}t} x(t) + \frac{\partial}{\partial y} f(x(t), y(t)) \frac{\mathrm{d}}{\mathrm{d}t} y(t) \\
&=
\end{aligned}
$$

Example: Envelope theorem

Suppose that f depends on a variable x and a parameter t. For a fixed t, let $x^*(t)$ be the value such that

$$f^*(t) := \min(\max)_x f(x,t) = f(x^*(t),t) \qquad \text{(the value function)},$$

Assume that such $x^*(t)$ exists, depends smoothly on t, and that it belongs to the interior of the domain of variation of x. In particular

$$\frac{\partial}{\partial x} f(x^*(t),t) = 0 \qquad \text{(first-order conditions)}$$

The chain rule yields the so-called Envelope Theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t} f^*(t) = \frac{\partial}{\partial x} f(x^*(t),t) \frac{\mathrm{d}}{\mathrm{d}t} x(t) + \frac{\partial}{\partial y} f(x^*(t),t) \frac{\mathrm{d}}{\mathrm{d}t} t = \frac{\partial}{\partial y} f(x^*(t),t)$$

$$\partial_{t_j} f^*(t) = \partial_{y_j} f(x^*(t), t) \qquad t = (t_1, \ldots, t_n) \in \mathbb{R}^n$$

Let $f(x, r_1, r_2) = x^{r_1} - r_2 x$ for $x \geq 0$, where $0 < r_1 < 1$.
This function is concave, and any solution of

$$\max_x f(x, r)$$

is positive, so a solution satisfies the first-order condition

$$r_1 x^{r_1 - 1} - r_2 = 0 \Rightarrow x^*(r) = (r_1/r_2)^{1/(1-r_1)},$$

So that the value function is

$$f^*(r) = (x^*(r))^{r_1} - r_2 x^*(r) = (r_1/r_2)^{r_1/(1-r_1)} - r_2 (r_1/r_2)^{1/(1-r_1)}$$

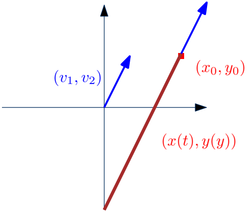
Hence

$$\begin{aligned} \frac{\partial}{\partial r_1} f^*(r) &= \frac{\partial f}{\partial r_1}(x^*(r), r) = (x^*(r))^{r_1} \ln(x^*(r)) \\ &= \frac{1}{1 - r_1} (r_1/r_2)^{r_1/(1-r_1)} \ln(r_1/r_2) \end{aligned}$$

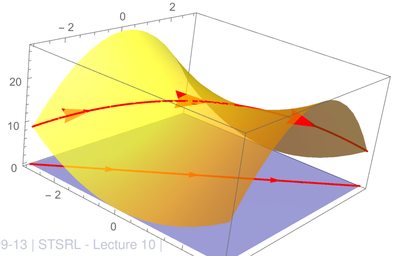
Example: Directional derivatives

Fix (v_1, v_2) a vector in the plane, and $(x_0, y_0) \in \mathbb{R}^2$ and let

$$\begin{cases} x(t) = x_0 + tv_1 \\ y(t) = y_0 + tv_2 \end{cases} \qquad t \in \mathbb{R}$$



This describe a line through the point (x_0, y_0) with direction (v_1, v_2).



So $f(x(t), y(t))$ gives the values of f along that line.

$$\begin{cases} x(t) = x_0 + tv_1 \\ y(t) = y_0 + tv_2 \end{cases}$$ $t \in \mathbb{R}$. By the chain rule

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t)) &= \frac{\partial}{\partial x} f(x(t), y(t)) v_1 + \frac{\partial}{\partial y} f(x(t), y(t)) v_2 \\ &= \begin{bmatrix} \frac{\partial}{\partial x} f(x(t), y(t)) & \frac{\partial}{\partial x} f(x(t), y(t)) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{aligned}$$

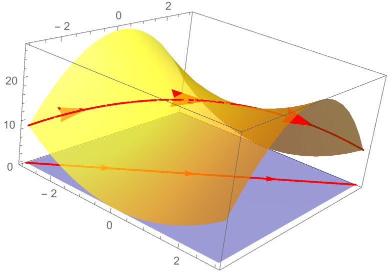
For $t = 0$

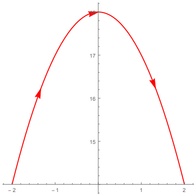
$$D_v f(x_0, y_0) := \frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t))|_{t=0} = \overbrace{\begin{bmatrix} \frac{\partial}{\partial x} f(x_0, y_0) & \frac{\partial}{\partial x} f(x_0, y_0) \end{bmatrix}}^{Df(x_0, y_0)} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$D_v f(x_0, y_0)$ is called the directional derivative of f in the direction v.
$Df(x_0, y_0)$ is called the total derivative of f at (x_0, y_0).
Note that $D_{e_1} f(x_0, y_0) = \frac{\partial}{\partial x} f(x_0, y_0)$, $D_{e_2} f(x_0, y_0) = \frac{\partial}{\partial y} f(x_0, y_0)$

Lets take in particular

$$f(x,y) = 18+x^2-2y^2, \quad \begin{cases} x(t) = t \\ y(t) = t \end{cases}, t \in \mathbb{R} \quad g(t) = f(t,t) = 18-t^2$$





Then $D_{(1,1)}f(0,0) = g'(0) = 0$.
Note that $D_{e_1}f(0,0) = D_{e_2}f(0,0) = 0$

Section 3
Unconstrained optimisation: Necessary conditions

Stationary points

We say that a point (x_0, y_0) is a stationary point (critical point) if

$$\frac{\partial}{\partial x} f(x_0, y_0) = 0 \qquad \text{AND} \qquad \frac{\partial}{\partial y} f(x_0, y_0) = 0,$$

i.e. $Df(x_0, y_0) = (0, 0)$.

Examples

The point $(0, 0)$ is a stationary point for the functions $f(x, y) = x^2 + y^2$ and $f(x, y) = x^2 - y^2$

Theorem

Given $f : D = [a, b] \to \mathbb{R}$, if x_0 lies inside the domain of definition D, and x_0 is a local extremum, then $f'(x_0) = 0$. That is, x_0 is a stationary point (critical point).

The set $[a, b]$ is closed (i.e. contain its boundary) and bounded (i.e. we can enclose it within a circle).
Sets that are closed and bounded are called compact.

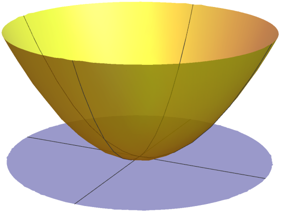
Theorem

Let $D \subset \mathbb{R}^2$ be a compact set. Given $f : D \to \mathbb{R}$, if (x_0, y_0) lies inside the domain of definition D, and (x_0, y_0) is a local extreme, then $Df(x_0, y_0) = (0, 0)$. That is, (x_0, y_0) is a stationary point (critical point).

Example

Let $f(x, y) = x^2 + y^2$ on the set of points D with $x^2 + y^2 \leq 1$.
The set D is a disk, centered at $(0, 0)$ of radius 1.
Note that

$$\begin{cases} \frac{\partial}{\partial x} f(x, y) = 2x = 0 \Leftrightarrow x = 0 \\ \frac{\partial}{\partial y} f(x, y) = 2y = 0 \Leftrightarrow y = 0 \end{cases}$$



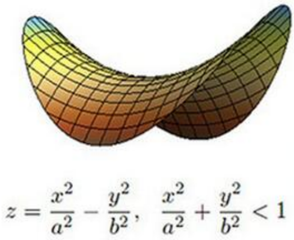
So $(0, 0)$ is the only stationary point.
Clearly or all (x, y),

$$f(x, y) = x^2 + y^2 \geq 0 = f(0, 0)$$

so $(0, 0)$ is a (local) minimum for f.

Not every stationary point is a local extreme.

Example The function $f(x, y) = x^2 - y^2$ defined on the set of points $x^2 + y^2 \leq 1$, satisfies that $(0, 0)$ is a stationary point inside the domain, but it is not an extreme point.



Thank you for your attention!