Lecture 9 Functions of two variables

A function f that depends on two variables say (x,y)

$$f : D \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto f(x,y)$$

Partial derivatives

We define the partial derivative of f with respect to x at (x_0,y_0) as

$$\frac{\partial f}{\partial x}(x_0,y_0) = \boxed{f'_x(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

We fix y_0 and differentiate f as a one-variable function of x.

Similarly

$$\frac{\partial f}{\partial y}(x_0,y_0) = f'_y(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0,y_0+h) - f(x_0,y_0)}{h}$$

ii) $f(x,y) = x^2+y^2$

$$\frac{\partial f}{\partial x}(x,y) = f'_x(x,y) = 2x+0 = 2x$$

$$\frac{\partial f}{\partial y}(x,y) = f'_y(x,y) = 0+2y = 2y$$

iii) $f(x,y) = (x^2+y)e^{x\cdot y}$

Calculate $\boxed{\frac{\partial f}{\partial x}(x,y)}$ & $\frac{\partial f}{\partial y}(x,y)$

$$f'_x(x,y) = \underbrace{\frac{\partial}{\partial x}(x^2+y)}_{2x}\cdot e^{x\cdot y} + (x^2+y)\underbrace{\frac{\partial}{\partial x}(e^{xy})}_{ye^{xy}}$$

$$\frac{d}{dx}(f(g(t))) = \frac{df}{dg}\cdot\frac{dg}{dt}$$

$$= [2x + (x^2+y)y]e^{xy} = [2x+x^2y+y^2]e^{xy}$$

$f''_{xx}(x,y)$

Higher order derivatives

One can define
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) =: \frac{\partial^2 f}{\partial x^2}(x,y) =: f^{(2)}_{xx}(x,y)$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = \frac{\partial^2 f}{\partial x \partial y}(x,y) =: f^{(2)}_{xy}(x,y)$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y) =: \frac{\partial^2 f}{\partial y \partial x}(x,y) =: f^{(2)}_{yx}(x,y)$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) =: \frac{\partial^2 f}{\partial y^2}(x,y) =: f^{(2)}_{yy}(x,y)$$

Example

$$\boxed{f(x,y) = x^3 e^{y^2}}$$

$$\begin{cases} f'_x(x,y) = \underline{3x^2 e^{y^2}} \\ f'_y(x,y) = 2yx^3 e^{y^2} \end{cases}$$

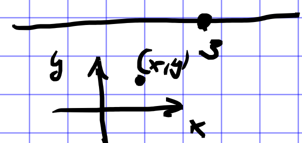
$$\begin{cases} f''_{xx}(x,y) = 6x e^{y^2} & f''_{yx}(x,y) = 6yx^2 e^{y^2} \\ f''_{xy}(x,y) = 6yx^2 e^{y^2} & f''_{yy}(x,y) = \dots = 2x^3 e^{y^2}(1+2y^2) \end{cases}$$

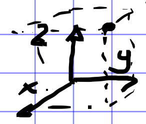
- $f''_{xx} = 3 \cdot 2x\, e^{y^2} = 6x e^{y^2}$
- $f''_{xy}(x,y) = (f'_y)'_x(x,y) = (2yx^3 e^{y^2})'_x = 6yx^2 e^{y^2}$

Theorem If the function f is "nice" then
$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)}$$

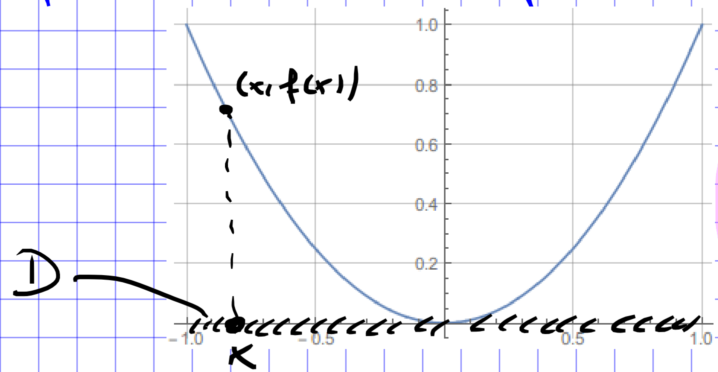
Geometric interpretation

Algebraic concept	Geometric concept
A real number x	A point on a line $\mathbb{R}$
A pair of numbers (x,y)	A point on a plane $\mathbb{R}^2$
A triple of number (x,y,z)	A point in space $\mathbb{R}^3$





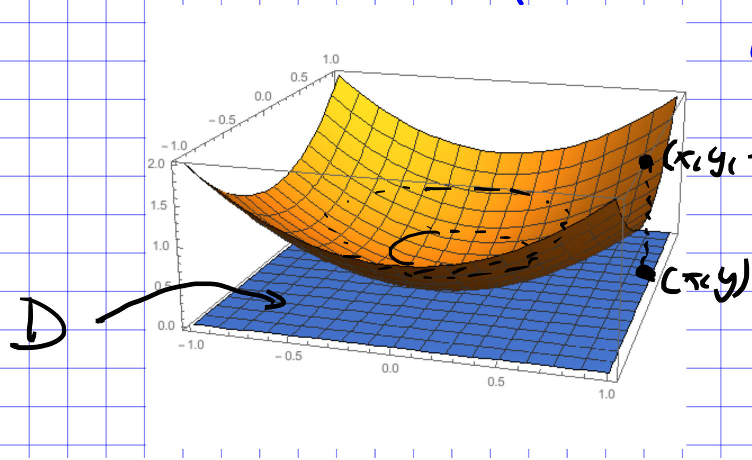
The graph of $f : D \to \mathbb{R}$ - if D is a subset of $\mathbb{R}$ $\quad (x, f(x)) \ \ x \in D$ (points in the plane)



$f(x) = x^2$

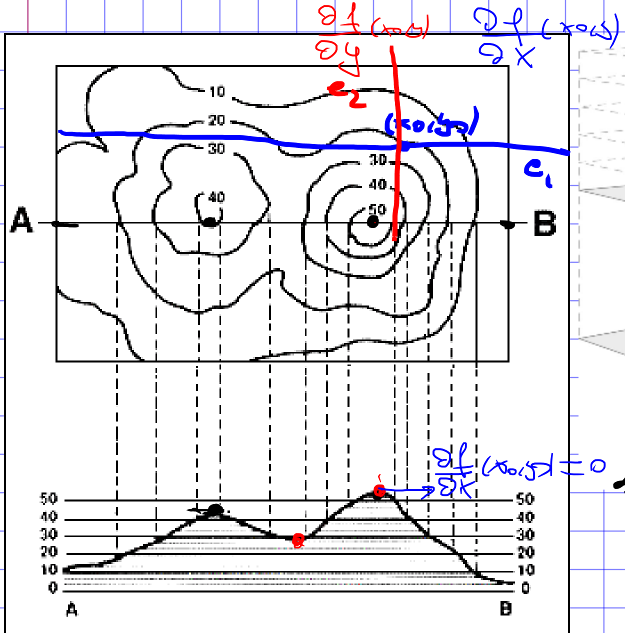
$-1 \leq x \leq 1$

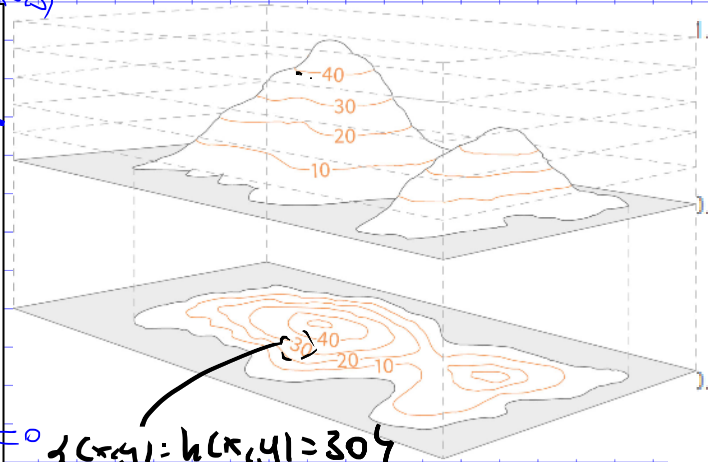
- if D is a subset of $\mathbb{R}^2$ $\quad (x, y, f(x,y)) \ \ (x,y) \in D$ (points in space)



$f(x,y) = x^2 + y^2$

$-1 \leq x \leq 1, \ -1 \leq y \leq 1$



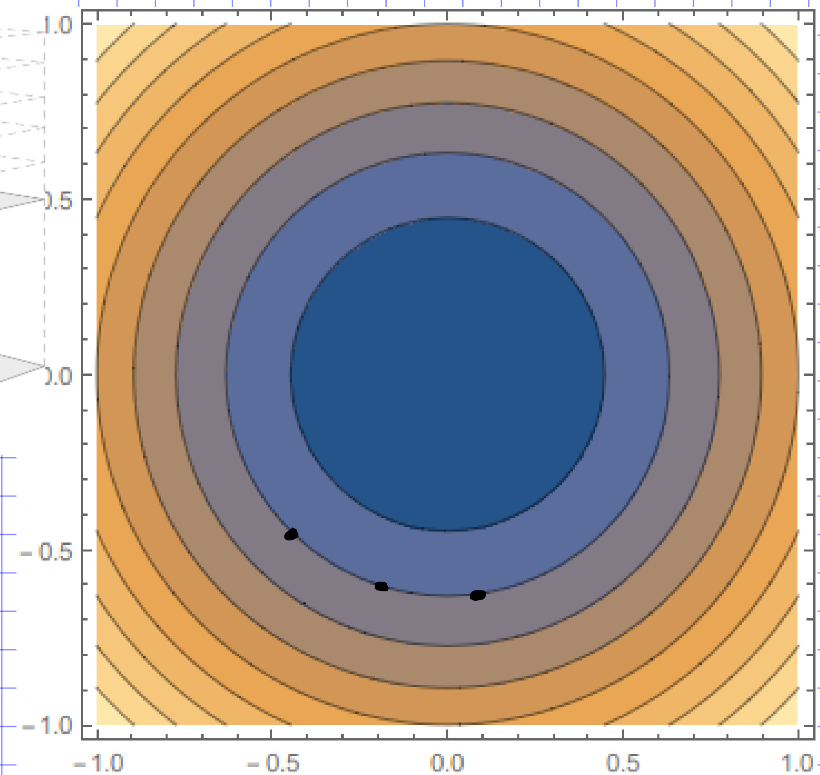


$\{(x,y) : h(x,y) = 30\}$

level curves

$h : D \longrightarrow \mathbb{R}$

$(x,y) \longmapsto h(x,y)$



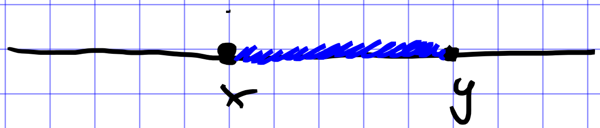
$[f(x,y) = x^2 + y^2]$

$x^2 + y^2 = c$

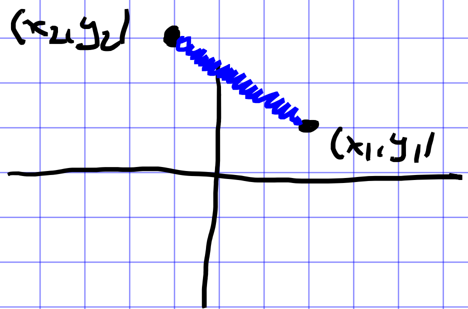
A way of representing graphically a function f is by plotting in the (x,y)-plane the set of points solving the equation

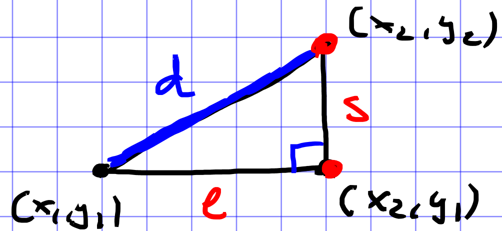
$$[f(x,y) = c] \quad \text{for a range of constants } c$$

These are called the <u>level curves of f</u>



$d(x,y) = |y-x|$





Pithagoras Theorem. $d^2 = e^2 + s^2$

$d = \sqrt{e^2 + s^2}$

$e = |x_2 - x_1|$

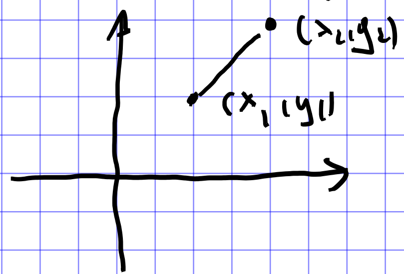
$s = |y_2 - y_1|$

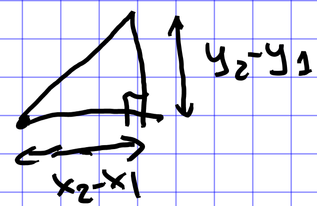
$$d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$(x_1,y_1,z_1),\ (x_2,y_2,z_2)$

$$d((x_1,y_1,z_1),(x_2,y_2,z_2)) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Distance between points of the plane





$$d((x_1,y_1),(x_2,y_2)) = \left((x_2-x_1)^2+(y_2-y_1)^2\right)^{\frac{1}{2}} \quad \text{(Pythagoras theorem)}$$

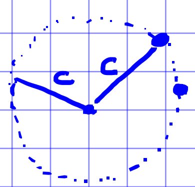
Similarly the distance between two points of $\mathbb{R}^3$ is defined as

$$d((x_1,y_1,z_1),(x_2,y_2,z_2)) = \left((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right)^{\frac{1}{2}}$$

The points $(x,y) \in \mathbb{R}^2$ such that

$$\boxed{x^2+y^2=c^2} \iff c = \sqrt{(x-0)^2+(y-0)^2}$$

is the points in the plane such that



$$d((x,y),(0,0)) = c$$

Which are the points of a circle of radius c

$$\frac{d}{dt} f(u(t)) = \frac{df}{du}(u(t)) \cdot \frac{du}{dt}(t)$$ Chain rule.

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot \frac{dy}{dt}$$

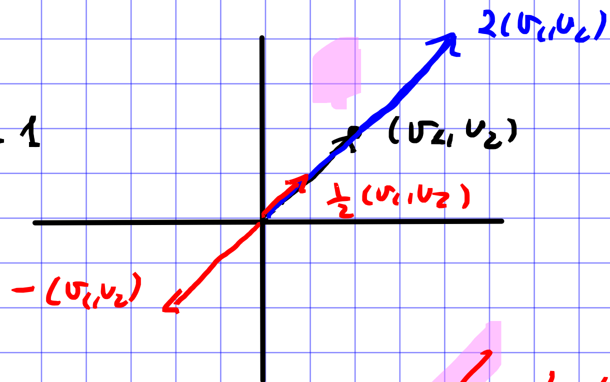
$f(x,y) = e^{x+y}$ [t belongs to the real numbers]

$$(x(t), y(t)) = (tv_1, tv_2) = t(v_1, v_2)$$

$$\begin{cases} x(t) = t v_1 \\ y(t) = t v_2 \end{cases} \qquad t \in \mathbb{R}$$

$$\frac{\partial f}{\partial x} = e^{x+y} \cdot 1$$

$$\frac{\partial f}{\partial y} = e^{x+y} \cdot 1$$



$t = 2 \to 2(v_1, v_2)$

$t = \frac{1}{2} \to \frac{1}{2}(v_1, v_2)$

$t = -1 \to -(v_1, v_2)$

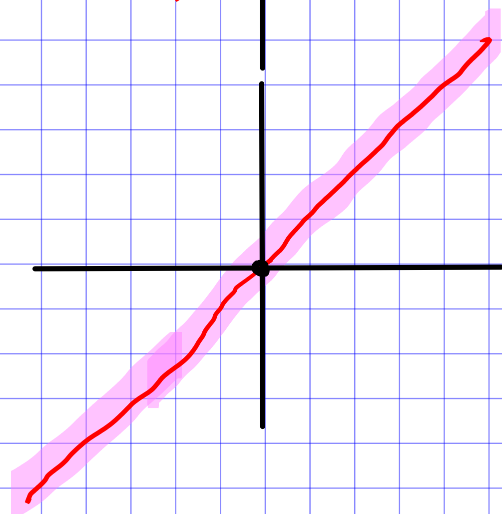
$$\frac{d}{dt}\big(f(x(t), y(t))\big) =$$

$$= \frac{\partial f}{\partial x}(x(t), y(t)) \cdot \frac{d}{dt} x(t)$$

$$+ \frac{\partial f}{\partial y}(x(t), y(t)) \cdot \frac{dy}{dt}$$

$$= e^{x(t)+y(t)} \cdot v_1 + e^{x(t)+y(t)} \cdot v_2$$

$$= e^{t(v_1+v_2)} (v_1 + v_2)$$



$t\,\underline{(v_1, v_2)}$ $\quad t \in \mathbb{R}$

Direction (v_1, v_2)

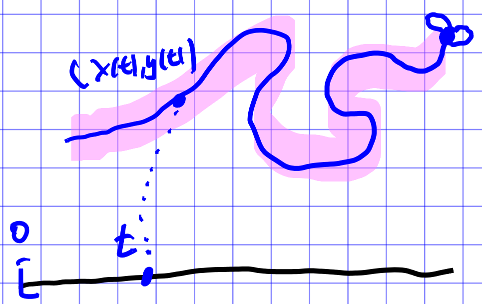
The Chain rule

Assume that $f(x,y)$ is a function of two variables and both $x = x(t)$ and $y = y(t)$.

So $f(x(t), y(t)) = g(t)$ is a function of one variable.

$$\frac{d}{dt} g(t) = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot \frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot \frac{dy}{dt}(t)$$

Note: the points $(x(t), y(t))$ can be thought as describing a location on the plane at a time t



Curve on the plane
$(x(t), y(t))$

Suppose $f(x,y) = x^2y + y^3$ $\longrightarrow f_x = 2xy \quad f_y = x^2+3y^2$

Suppose that $x = x(t) = t^2$ $\longrightarrow x'(t) = 2t$

$y = y(t) = t^3$ $\longrightarrow y'(t) = 3t^2$

Then $f(x(t), y(t)) = (t^2)^2 t^3 + (t^3)^3 = t^7 + t^9 = g(t)$

$g'(t) = 7t^6 + 9t^8$

Also

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}(x(t),y(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(x(t),y(t)) \cdot y'(t)$$

$$= 2xy \cdot (2t) + (x^2+3y^2)(3t^2)$$

$$= 2t^2t^3\,(2t) + ((t^2)^2+3(t^3)^2)\,(3t^2)$$

$$= 4t^6 + 3(t^6+3t^8) = 7t^6+9t^8$$

$$\begin{pmatrix} a & b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} := a \cdot x + b \cdot y \qquad \text{(scalar product)}$$

$$\begin{pmatrix} a & b & c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \cdot x + b \cdot y + c \cdot z$$

$$\begin{pmatrix} 1, & 2, & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \cdot x + 2 \cdot y + 3 \cdot z = x + 2y + 3z$$

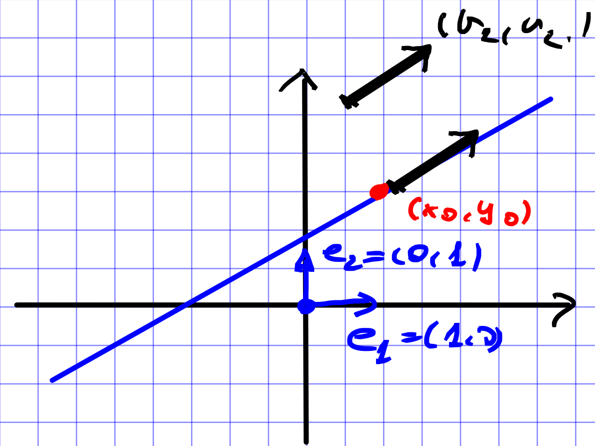
$$\underline{\quad} \cdot \underline{\quad} \cdot \underline{\quad} \cdot \underline{\quad}$$

$$2a + 3b = 1 \iff \begin{pmatrix} 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 1$$

Special case: Directional derivatives

Let $\vec{u} = (u_x, u_y)$ be a vector and (x_0, y_0) be a point in the domain of definition of $f(x,y)$

Let $\begin{cases} x(t) = x_0 + t\, v_1 \\ y(t) = y_0 + t\, v_2 \end{cases}$ $t \in \mathbb{R}$.



$(x(t), y(t))$ parametrises a straight line.

The chain rule gives

$$\frac{d}{dt}\left(f(x(t),y(t))\right)\Big|_{t=0} = \overbrace{\frac{\partial f}{\partial x}(x_0,y_0)\cdot \underbrace{x'(0)}_{v_1} + \frac{\partial f}{\partial y}(x_0,y_0)\cdot \underbrace{y'(0)}_{v_2}}$$

$\vec{v} = (v_1, v_2)$

$$= \underbrace{\left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right)\cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{\text{(Scalar Product.)}} = Df \cdot \vec{v}$$

$D_{\vec{v}} f$

$\shortparallel$

$\frac{d}{dt}\left(f(x(t),y(t))\right)\Big|_{t=0}$ is called the directional derivative of f in the direction $\vec{v}$

$$\left[\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) \equiv Df(x,y)\right]$$

Scalar product

$$(a, b)\begin{pmatrix} c \\ d \end{pmatrix} = a\cdot c + b\cdot d$$

$Df \cdot \vec{v}$

If $\underline{\vec{e}_1 = (1,0)}$ then

$$D_{e_1} f = \left(\frac{\partial f}{\partial x}(x_0, y_0) , \frac{\partial f}{\partial y}(x_0, y_0) \right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\frac{\partial f}{\partial x}(x_0, y_0)}$$

and similarly

$$\boxed{D_{e_2} f = \frac{\partial f}{\partial y}(x_0, y_0)}$$

We say that a point (x_0, y_0) is a critical point (stationary point) if

$$\boxed{\frac{\partial f}{\partial x}(x_0, y_0) = 0 \quad \underline{\underline{\text{and}}} \quad \frac{\partial f}{\partial y}(x_0, y_0) = 0}$$

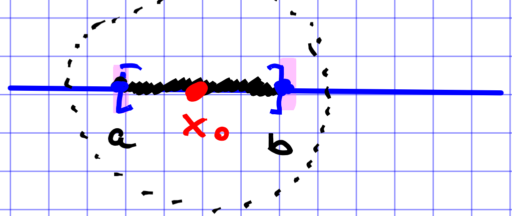
that is $\boxed{Df(x_0, y_0) = (0, 0)}$

(Recall) **Theorem** If $f : D = [a,b] \to \mathbb{R}$, x_0 lies inside the domain of definition D, and x_0 is a local maximum/minimum, then $f'(x_0) = 0$ (i.e. x_0 is a stationary point)

$[a,b]$ ≡ is a $\boxed{\text{closed}}$ and $\underbrace{\text{bounded (finite)}}$ interval } ≡ compact interval.

It contains its boundary

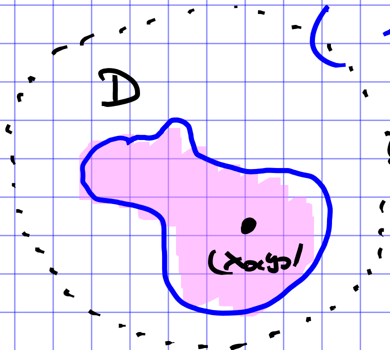
We can enclose it within a circle



Theorem Let D be a "compact*" set of $\mathbb{R}^2$

If $f : D \to \mathbb{R}$, (x_0, y_0) lies inside the domain of definition D, and (x_0, y_0) is a local maximum/minimum, then $Df(x_0) = \vec{0}$

(i.e. (x_0, y_0) is a stationary point)

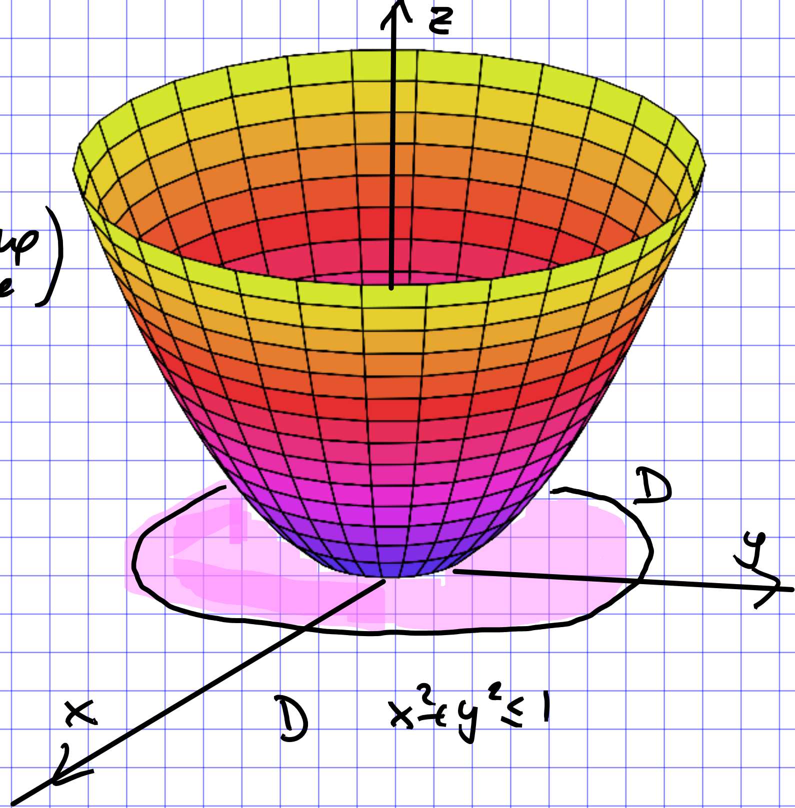


(*) D is closed (contains its boundary) and bounded (we can enclose it within a circle

Example

1) $f(x,y) = x^2+y^2-10$ on $x^2+y^2 \leq 1$

(disc of radius one in the plane)



Clearly

$$\boxed{f(x,y) = \underbrace{x^2+y^2}_{\geq 0} - 10 \geq -10 = f(0,0)}$$

So $(0,0)$ is a local minimum

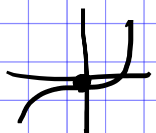
(indeed is global for f on $\mathbb{R}^2$)

Note

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 2x = 0 \iff x = 0 \\[2ex] \dfrac{\partial f}{\partial y}(x,y) = 2y = 0 \iff y = 0 \end{cases}$$

So $(0,0)$ is a <u>critical point</u>.

Q: How can we know whether is a maximum / minimum without plotting?



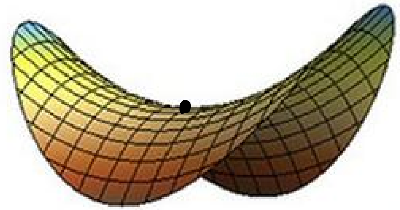
$f(x) = x^3$

$f'(x) = 3x^2 = 0 \iff x = 0$

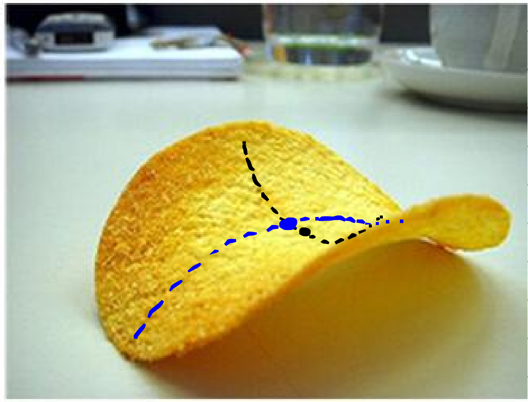
As it happens for functions on one variable, not every stationary point is a local maximum / minimum

$[f(x,y) = x^2 - y^2]$

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 2x = 0 \iff x=0 \\ \dfrac{\partial f}{\partial y}(x,y) = -2y = 0 \iff x=0 \end{cases}$$



$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$$



BUT

Is not an extreme point

For functions in one variable, we learned how to determine this by looking at the sign of the derivative to the left / right of the critical point. But the situation does not look so simple in two variables

Second derivative test for functions in one variable. (Assume that $c=0$)

THEOREM 9.6.2 (SECOND-DERIVATIVE TEST FOR LOCAL EXTREMA)

Let f be a twice differentiable function in an interval I, and let c be an interior point of I.

(i) If $f'(c) = 0$ and $\boxed{f''(c) < 0}$, then $x = c$ is a strict local maximum point.

(ii) If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a strict local minimum point.

(iii) If $f'(c) = 0$ and $f''(c) = 0$, then x could be a local maximum, a local minimum, or neither.

Idea of the proof: Taylor's formula gives ~~$f(x)$~~

$$\underbrace{f(x) \approx f(0) + \frac{f''(0)}{2} x^2}_{\text{Parabola.}}$$

Recall

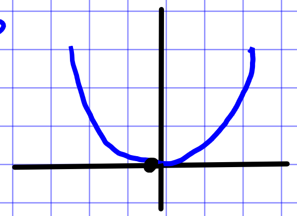
i) & ii) $f(x) = ax^2 + b$

$f'(x) = 2ax = 0$

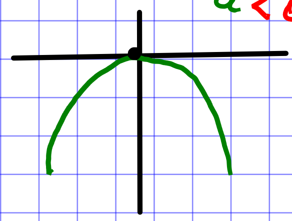
$\Leftrightarrow x = 0$

$f''(0) = 2a$

$a > 0$



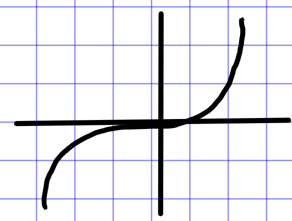
$a < 0$



iii) $f(x) = x^3$

$f'(x) = 3x^2$

$f''(0) = 0$



THEOREM 9.2.2 (EXTREMA OF CONCAVE AND CONVEX FUNCTIONS)

Suppose that f is a function defined in an interval I and that c is a critical point for f in the interior of I.

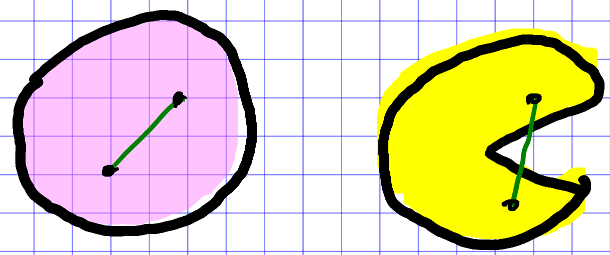
(i) If f is concave, then c is a maximum point for f in I.

(ii) If f is convex, then c is a minimum point for f in I.

- $f''(x) < 0$ for all $x \in I$
- $f''(x) > 0$ for all $x \in I$

Second derivative test for functions in two variables

Note: A set $S \subset \mathbb{R}^2$ is convex if for every two points $\bar{x}, \bar{y}$ in S the entire interval from $\bar{x}$ to $\bar{y}$ is contained in S



Convex   Not convex

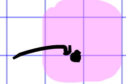
Given a function of two variables we define its Hessian matrix as

$$H(x,y) = \begin{pmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{pmatrix}$$

(2 rows & 2 columns) matrix.

and we define its determinant as

$$\left[\det(H(x,y)) = f''_{xx} \cdot f''_{yy} - (f_{xy})^2\right]$$

$\frac{\partial f}{\partial y}(x_0,y_0)=0$ (x_0,y_0)  S $\frac{\partial f}{\partial x}(x_0,y_0)=0$

Theorem

Suppose that S is a convex domain, (x_0,y_0) is an interior critical point of f.

Assume that $\det(H(x,y)) = f_{xx}f_{yy} - (f_{xy})^2 \geq 0$ for all $(x,y) \in S$

i) If $f''_{xx}(x,y) \leq 0$, $f''_{yy}(x,y) \leq 0$ for all (x,y)

then f is concave* on S and

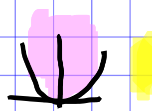
(x_0,y_0) is a maximum point for f in S

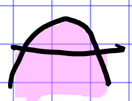
i) If $f''_{xx}(x,y) \geq 0$, $f''_{yy}(x,y) \geq 0$

then f is convex* on S and

(x_0,y_0) is a minimum point for f in S

(*)

 A function is <u>convex</u> if the region above its graph is convex

 A function is concave if the region below its graph is convex

Example

$$f(x,y) = 2x^2 + 4xy + 3y^2 - x - 2y$$

Critical points

$$\begin{cases} f'_x = 4x + 4y - 1 = 0 \\ f'_y = 4x + 6y - 2 = 0 \end{cases} \iff \begin{cases} 4x + 4y = 1 \\ 4x + 6y = 2 \end{cases}$$

We can find that the solution to this system of equations is given by (Check it!)

$$\left[x = -\tfrac{1}{4} \quad y = \tfrac{1}{2} \right]$$ (Soon we will learn how to solve this system!)

Now

$$H(x,y) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 6 \end{pmatrix}$$

Notice that $f''_{xx} = 4 > 0$, $f''_{yy} = 6 > 0$

and $\det(H(x,y)) = 4 \cdot 6 - 4^2 = 24 - 16 = 8 > 0$

So we can say that f is convex and has a minimum at $\left(-\frac{1}{4}, \frac{1}{2}\right)$

Second derivative test for Local extrema.

Let (x_0,y_0) be a critical point of f (i.e. $Df(x_0,y_0)=(0,0)$)

Recall: $\det(H(x_0,y_0)) = f''_{xx}(x_0,y_0)\cdot f''_{yy}(x_0,y_0) - \left(f''_{yx}(x_0,y_0)\right)^2$

i) If $\det(H(x_0,y_0)) > 0$ and $f''_{xx}(x_0,y_0) > 0$

then f has a local minimum at (x_0,y_0)

ii) If $\det(H(x_0,y_0)) > 0$ and $f''_{xx}(x_0,y_0) < 0$

then f has a local maximum at (x_0,y_0)

iii) If $\det(H(x_0,y_0)) < 0$ then (x_0,y_0) is

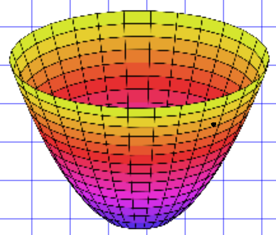
then f has a saddle point at (x_0,y_0)

Idea

$f(x,y) = x^2 + y^2$

$H(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

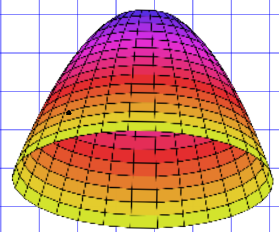
$\det(H) = 4 > 0$



$f(x,y) = -(x^2+y^2)$

$H(x,y) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

$\det(H) = 4 > 0$



$f(x,y) = x^2 - y^2$

$H(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$\det(H) = -4 < 0$