Lecture 10

If a function $f: D \to \mathbb{R}$ has local extrema at the point (x_0, y_0) and it's an interior point of D

then $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$ (i.e. (x_0, y_0) is a stationary point)

Second derivative test for functions in one variable. (Assume that $c=0$)

THEOREM 9.6.2 (SECOND-DERIVATIVE TEST FOR LOCAL EXTREMA)

Let f be a twice differentiable function in an interval I, and let c be an interior point of I.

(i) If $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a strict local maximum point.

(ii) If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a strict local minimum point.

(iii) If $f'(c) = 0$ and $f''(c) = 0$, then x could be a local maximum, a local minimum, or neither.

Idea of the proof: Taylor's formula gives ~~flex~~

$$f(x) \simeq \underbrace{f(0) + \frac{f''(0)}{2} x^2}_{\text{Parabola.}}$$

Recall

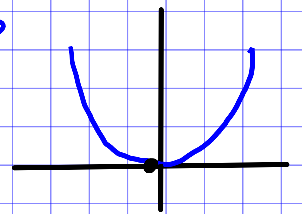
i) & ii) $f(x) = ax^2 + b$

$f'(x) = 2ax = 0$

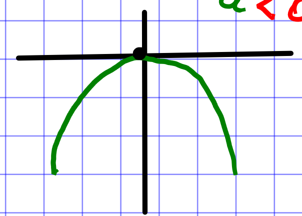
$\Leftrightarrow x = 0$

$f''(0) = 2a$

$a > 0$



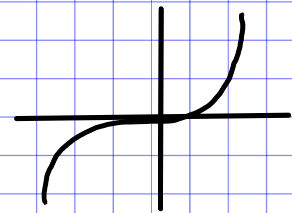
$a < 0$



iii) $f(x) = x^3$

$f'(x) = 3x^2$

$f''(0) = 0$



THEOREM 9.2.2 (EXTREMA OF CONCAVE AND CONVEX FUNCTIONS)

Suppose that f is a function defined in an interval I and that c is a critical point for f in the interior of I.

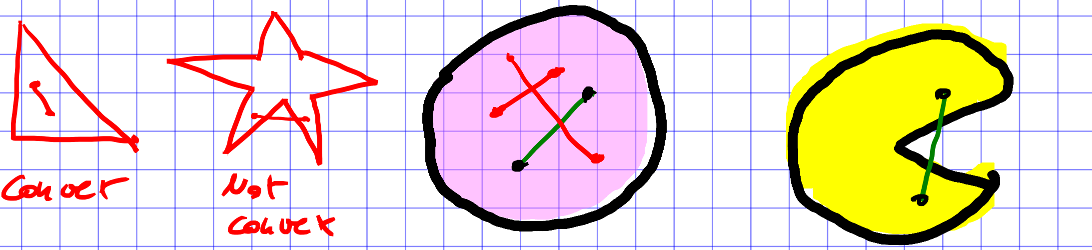
(i) If f is concave, then c is a maximum point for f in I.

(ii) If f is convex, then c is a minimum point for f in I.

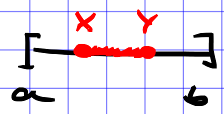
- $f''(x) < 0$ for all $x \in I$
- $f''(x) > 0$ for all $x \in I$

Second derivative test for functions in two variables

Note: A set $S \subset \mathbb{R}^2$ is convex if for every two points $\bar{x}, \bar{y}$ in S the entire interval from $\bar{x}$ to $\bar{y}$ is contained in S



Convex Not convex



· [a,b] är konvex

Given a function of two variables we define its Hessian matrix as

$$H(x,y) = \begin{pmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{pmatrix}$$

(2 rows & 2 columns) matrix.

and we define its determinant as

$$\det(H(x,y)) = f''_{xx} \cdot f''_{yy} - (f''_{xy})^2$$

$\frac{\partial f}{\partial y}(x_0,y_0)=0$ (x_0,y_0) S $\frac{\partial f}{\partial x}(x_0,y_0)=0$

Theorem

Suppose that S is a convex domain, (x_0,y_0) is an interior critical point of f

Assume that $\det(H(x,y)) = f_{xx} f_{yy} - (f_{xy})^2 \geq 0$ for all $(x,y) \in S$

i) If $f''_{xx}(x,y) \leq 0$, $f''_{yy}(x,y) \leq 0$ for all (x,y)

then f is concave* on S and

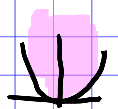
(x_0,y_0) is a maximum point for f in S

i) If $f''_{xx}(x,y) \geq 0$, $f''_{yy}(x,y) \geq 0$

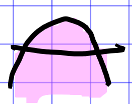
then f is convex* on S and

(x_0,y_0) is a minimum point for f in S

(*)



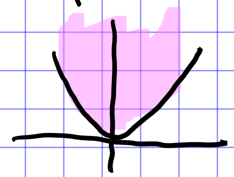
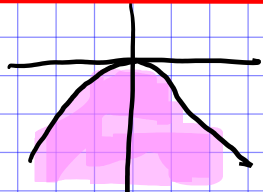
A function is <u>convex</u> if the region above its graph i convex



A function is concave if the region below its graph i convex

How can one recall this theorem?

1-D case (Model for convex / concave functions)

$f(x) = x^2$	$f(x) = -x^2$
	
$x = 0$ is a stationary point ($f'(0) = 0$)	
$f''(0) > 0$	$f''(0) < 0$
$f''(x) \geq 0 \equiv$ Convex	$f''(x) \leq 0 \equiv$ Concave.

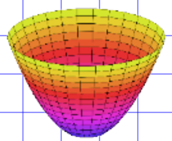
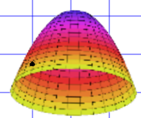
THEOREM 9.2.2 (EXTREMA OF CONCAVE AND CONVEX FUNCTIONS)

Suppose that f is a function defined in an interval I and that c is a critical point for f in the interior of I.

(i) If f is concave, then c is a maximum point for f in I.

(ii) If f is convex, then c is a minimum point for f in I.

2D

$f(x,y) = x^2 + y^2$ (*)	$f(x,y) = -(x^2+y^2)$
$(x,y) = (0,0)$ is a stationary point	
 ($S = \mathbb{R}^2$)	
$H(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$H(x,y) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$
$\det(H) = 4 > 0$	$\det(H) = 4 > 0$
$f_{xx}(x,y) \geq 0$ $f_{yy}(x,y) \geq 0$	$f_{xx}(x,y) \leq 0$ $f_{yy}(x,y) \leq 0$
$\det H(x,y) = f_{xx} f_{yy} - (f_{xy})^2 \geq 0$	$\det H(x,y) = f_{xx} f_{yy} - (f_{xy})^2 \geq 0$
Convex	Concave

THEOREM 17.2.1 (CONCAVITY OR CONVEXITY AS A SUFFICIENT CONDITION)

Suppose that the function $z = f(x, y)$ of two variables is defined on a convex domain S, and that (x^0, y^0) is an interior critical point in S. Then:

(a) in case the function f is concave, the point (x^0, y^0) is a maximum point;

(b) in case the function f is convex, the point (x^0, y^0) is a minimum point.

(*) $\begin{cases} f'_x = 2x & f'_y = 2y \\ f''_{xx} = 2 & f''_{xy} = 0 \quad f''_{yy} = 2 \end{cases}$

$H(x,y) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}$

$\det(H) = f''_{xx} \cdot f''_{yy} - (f''_{yx})^2$

Example

$$f(x,y) = 2x^2 + 4xy + 3y^2 - x - 2y$$

Critical points

$$\begin{cases} f'_x = 4x + 4y - 1 = 0 \\ f'_y = 4x + 6y - 2 = 0 \end{cases} \iff \begin{cases} 4x + 4y = 1 \\ 4x + 6y = 2 \end{cases}$$

We can find that the solution to this system of equations is given by (Check it!)

$$\left[x = -\tfrac{1}{4} \quad y = \tfrac{1}{2} \right]$$ (Soon we will learn how to solve this system!)

Now

$$H(x,y) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 6 \end{pmatrix}$$

Notice that $f''_{xx} = 4 > 0$, $f''_{yy} = 6 > 0$

and $\det(H(x,y)) = 4 \cdot 6 - 4^2 = 24 - 16 = 8 > 0$

So we can say that f is convex and has a minimum at $\left(-\frac{1}{4}, \frac{1}{2}\right)$

Second derivative test for Local extrema.

Let (x_0, y_0) be a critical point of f (i.e. $Df(x_0,y_0) = (0,0)$)

Recall: $\det(H(x_0,y_0)) = f''_{xx}(x_0,y_0) \cdot f''_{yy}(x_0,y_0) - \left(f''_{yx}(x_0,y_0)\right)^2$

i) If $\det(H(x_0,y_0)) > 0$ and $f''_{xx}(x_0,y_0) > 0$

then f has a local minimum at (x_0,y_0)

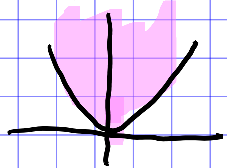
ii) If $\det(H(x_0,y_0)) > 0$ and $f''_{xx}(x_0,y_0) < 0$

then f has a local maximum at (x_0,y_0)

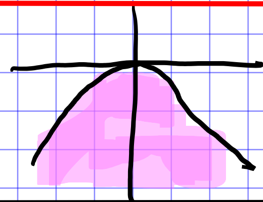
iii) If $\det(H(x_0,y_0)) < 0$ then (x_0,y_0) is

then f has a saddle point at (x_0,y_0)

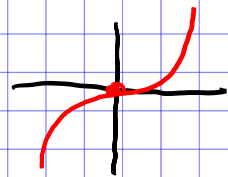
$f(x) = x^2$



$f(x) = -x$



$f(x) = x^3$



$x = 0$ is a stationary point ($f'(0) = 0$)

$f''(0) > 0$

local minimum

$f''(0) < 0$

local maximum.

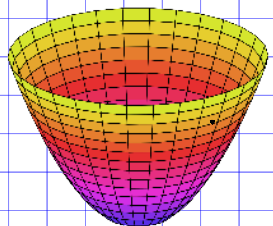
$f''(0) = 0$

$\begin{pmatrix} f''(x) < 0 & \text{if } x < 0 \\ f''(x) > 0 & \text{if } x > 0 \end{pmatrix}$ (Inflection point)

$f(x,y) = x^2 + y^2$

$H(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\det(H) = 4 > 0$

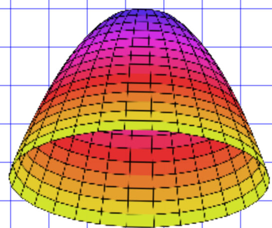


local minimum

$f(x,y) = -(x^2 + y^2)$

$H(x,y) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

$\det(H) = 4 > 0$

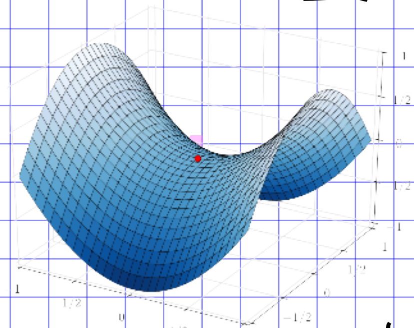


local maximum.

$f(x,y) = x^2 - y^2$

$H(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$(\det(H) = \underline{-4} < 0)$



Saddle point.

Let $F(x,y) = x^4 + y^4 - 36xy$. Find all stationary points for this function and determine whether they are local maximums, minimums, or saddle points. (2015)

Note that

$$\begin{cases} F'_x = 4x^3 - 36y = 0 \\ F'_y = 4y^3 - 36x = 0 \end{cases} \iff \begin{cases} y = \frac{x^3}{9} & (1) \\ x = y^3/9 & (2) \end{cases}$$

Using (1) in (2), any stationary point (x,y) satisfies that

$$x = \left(\frac{x^3}{9}\right)^3 \frac{1}{9} = \frac{x^9}{9^4} \iff x^9 - 9^4 x = 0$$

$$\iff \underbrace{x(x^8 - 3^8) = 0}_{9=3^2}$$

$$\iff x = 0 \text{ or } [x^8 = 3^8]$$

$$\iff x = 0 \text{ or } x = \pm\sqrt[8]{3^8} = \pm 3$$

If $x=0$, by (1), $y = \frac{0^3}{9} = 0$; If $x=-3$ then $y = \frac{(-3)^3}{3^2} = -3$.

If $x = 3$, by (1), $y = \frac{3^3}{3^2} = 3$

So there are 3 stationary points

$P_1 = (0,0)$ $P_2 = (3,3)$ and $P_3 = (-3,-3)$

Recall that

$$\begin{cases} F'_x = 4x^3 - 36y \\ F'_y = 4y^3 - 36x \end{cases} \quad \text{then} \quad \begin{cases} F''_{xx} = 12x^2 \\ F''_{xy} = F''_{yx} = -36 \\ F''_{yy} = 12y^2 \end{cases}$$

$(F'_x)'_y = (F'_y)'_x$

So the Hessian matrix becomes

$$H(x,y) = \begin{pmatrix} 12x^2 & -36 \\ -36 & 12y^2 \end{pmatrix}$$ and its determinant

$12 = 2^2 \cdot 3$ $36 = 2^2 \cdot 3^2$

$$\left(D = \det(H(x,y)) = 12^2 x^2 y^2 - (-36)^2 = 2^4 \cdot 3^2 x^2 \cdot y^2 - 2^4 \cdot 3^4 \right.$$

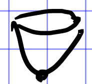
$$\left. = 2^4 \cdot 3^2 (x^2 y^2 - 3^2) \right)$$

$$\begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{cases} F_{xx} = 12x^2 \geq 0 \\ D(x,y) = 2^4 \cdot 3^2 \, (x^2y^2 - 3^2) = 2^4 \cdot 3^2 \, ((xy)^2 - 3^2) \end{cases}$$

Critical point	$D(x,y)$	F''_{xx}	Type.
$(x,y) = (0,0)$	$2^4 \cdot 3^2 \, (0 - 3^2) = -2^4 \cdot 3^4 < 0$		$\boxed{\text{Saddle point.}}$
$(3,3)$	$2^4 \cdot 3^2 \, ((3^2)^2 - 3^2) = 2^4 \cdot 3^4 \, (3^2 - 1) > 0$	$12 \cdot 3^2 > 0$	Local minimum
$(-3,-3)$	$2^4 \cdot 3^4 \, (3^2 - 1) > 0$	$12 \cdot (-3)^2 > 0$	Local minimum.

$f(x,y) = x^2 + y^2$

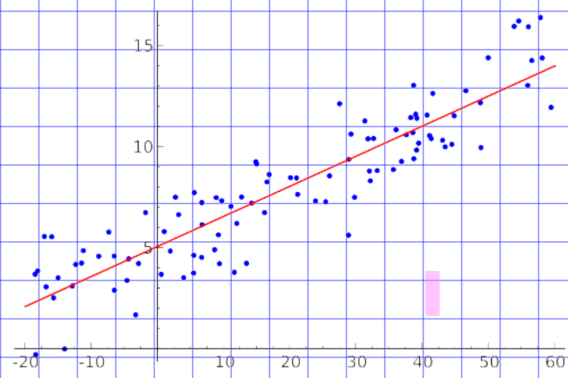


$H(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\det(H) > 0$

Suppose we have a batch of data

$$(x_t, y_t) \qquad [t = 1, \dots, T]$$



and we want to find a line

$$y = \alpha + \beta x$$

that approximates well those data

If $\quad y_t - \alpha - \beta x_t = e_t \quad$ is the error term

we want to minimise

$$L(\alpha, \beta) = \frac{1}{T} \sum_{t=1}^{n} e_t^2 \qquad \left[\sum_{j=1}^{n} a_j = a_1 + a_2 + \dots + a_n \right]$$

Expanding the square, and defining

$(\sigma_{xx} > 0)$

$\sigma_{xx} = 0$

$(x_1 = x_2 = x_3 \dots)$

$$\begin{cases} \sigma_{xx} = \frac{1}{T} \sum (x_t - \mu_x)^2 & \text{Variance} \\ \sigma_{yy} = \frac{1}{T} \sum (y_t - \mu_y)^2 & \\ \sigma_{xy} = \frac{1}{T} \sum (x_t - \mu_x)(y_t - \mu_y) & \text{Co-variance} \end{cases}$$

$$\boxed{\begin{aligned} \mu_x &= \frac{1}{T} \sum x_t \\ \mu_y &= \frac{1}{T} \sum y_t \\ &\text{(Arithmetic mean)} \end{aligned}}$$

One can show that

(quadratic function) $\rightsquigarrow (\alpha^2, \beta^2)$

$$L(\alpha, \beta) = \alpha^2 + \mu_y^2 + \beta^2\mu_x^2 - 2\alpha\mu_y - 2\beta\mu_x\mu_y + 2\alpha\beta\mu_x + \beta^2\sigma_{xx} - 2\beta\sigma_{xy} + \sigma_{yy}$$

$$\boxed{\begin{aligned} L'_\alpha &= 0 \\ L'_\beta &= 0 \end{aligned}}$$

One can then see that the unique critical point of $L(\alpha, \beta)$ occurs at

$$\begin{cases} \hat{\alpha} = \mu_y - \dfrac{\sigma_{xy}}{\sigma_{xx}}\mu_x \\[2ex] \hat{\beta} = \dfrac{\sigma_{xy}}{\sigma_{xx}} \end{cases}$$

Furthermor

- $$\begin{pmatrix} L''_{xx} & L''_{yx} \\ L''_{xy} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 2\mu_x \\ 2\mu_x & 2\sigma_{xx} \end{pmatrix}$$

and

$$L''_{xx}L''_{yy} - (L''_{xy})^2 = 4\sigma_{xx}.$$

Under the assumption that $\sigma_{xx} > 0$ we have that

1) L is convex
2) L attains its minimum at $(\hat{\alpha}, \hat{\beta})$

(See § 17.4 for the details)

The extreme Value theorem

Let $f: [a,b] \to \mathbb{R}$ be differentiable
($[a,b]$ closed and bounded)

i) Then there exist $x_m, x_n \in [a,b]$ such that

$$f(x_m) \le f(x) \le f(x_n) \quad \text{for all } x \in [a,b].$$

ii) The maximum/minimum are either

a) A critical point inside $[a,b]$

b) A border point (either a or b)

Let $f: D \to \mathbb{R}$ be differentiable
D is closed and bounded

i) Then there exists (x_m, y_m), (x_n, y_n) in D such that

$$f(x_m, y_m) \le f(x,y) \le f(x_n, y_n) \quad \text{for all } (x,y) \in D$$

ii) The maximum/minimum are either

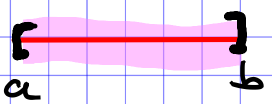
a) A critical point inside D

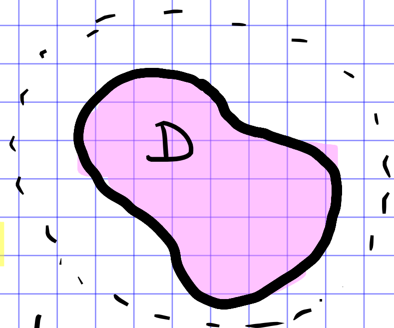
b) A border point

1) Make a list consisting of

▷ Critical points (in D)

▷ largest/smallest "Boundary points"





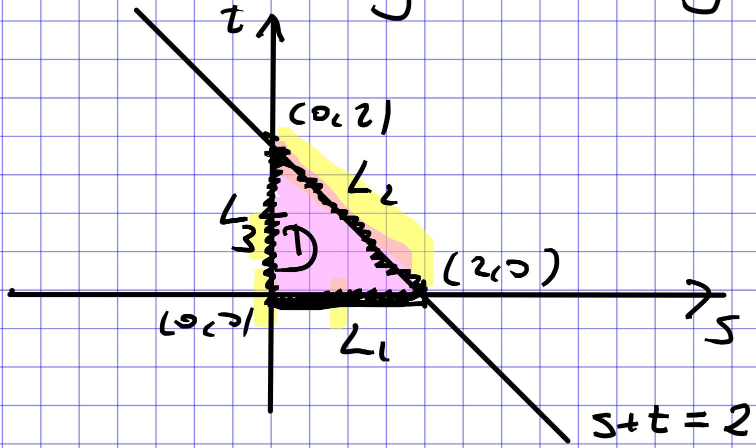
2) Evaluate f at all the points in the list

3) Find the largest and smallest value

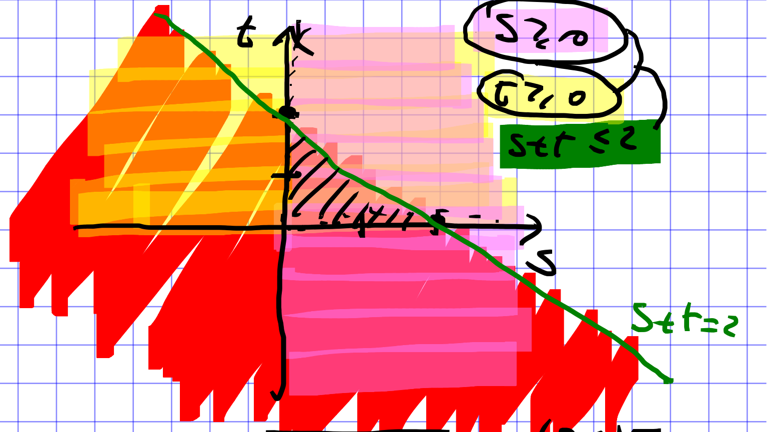
One of the difficulties consists of parametrizing (describing) the boundary of D

6. A quantity C is determined by inputs s, t according to the formula $C = st(s+t-1)$. Find the maximum and minimum possible values for C, subject to the constraints $s \geq 0$, $t \geq 0$, and $s + t \leq 2$. (2018)

Lets start by visualising the domain D



- $s \geq 0$
- $t \geq 0$
- $s+t \leq 2$



$\boxed{s+t=2}$ → (0,2), → (2,0)

$\boxed{t = 2-s}$

The border of D consists of three pieces L_1, L_2 and L_3

1) L_1 is the set of points (s,t) such that $[t=0]$ and $(s \in [0,2])$

3) L_3 is the set of points (s,t) such that $s=0$ and $t \in [0,2]$

2) L_2 is the set of points (s,t) such that $s+t=2$, $s \in [0,2]$
$(t = 2-s)$

$$\left[C(s,t) = st\,(s+t-1) \right]$$

- stationary points

$$\begin{cases} C'_s(s,t) = t\,(s+t-1) + st = t\,(2s+t-1) \\ C'_t(s,t) = s\,(2t+s-1) \end{cases}$$

Note

- $C'_s = 0 \iff t=0$ or $2s+t=1$
- $C'_t = 0 \iff s=0$ or $2t+s=1$

We have that the critical points are (s,t) satisfying

- $t=0$ & $s=0$ i.e. $(0,0)$
- $t=0$ & $2\cdot 0+s=1$ i.e. $(1,0)$
- $s=0$ & $2\cdot 0+t=1$ i.e. $(0,1)$
- $\begin{cases} 2s+t=1 \\ s+2t=1 \end{cases} \iff \begin{cases} 2s+t=1 \\ 2s+4t=2 \end{cases} \overset{(2)-(1)}{\iff} \begin{cases} 2s+t=1 \\ 0s+3t=1 \end{cases} \iff \begin{cases} 2s=1-t \\ t=1/3 \end{cases}$

$$\iff \begin{cases} s=(1-\frac{1}{3})/2 = \frac{1}{3} \\ t=\frac{1}{3} \end{cases}$$

That is $(1/3, 1/3)$

Note that only $(1/3, 1/3)$ lie inside D

▶ Border: We look for the maximum and minimum values of the function when is restricted to the border

$$C(s,t) = st\,(s+t-1)$$

▶ Along the side L_1 $(t=0,\ s\in[0,2])$ the function

$$C(s,t) = C(s,0) = 0$$

(The function is constant along L_1)

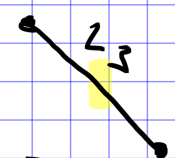
▶ Along the side L_3 $(s=0,\ t\in[0,2])$ the function

$$C(s,t) = C(0,t) = 0$$

(The function is constant along L_3)

▶ Along the side L_2 $(s\in[0,2],\ t=2-s)$

$$x(s) = s \qquad y(s) = 2-s.$$



$$C(s,t) = C(x(s), y(s)) = s(2-s)(s+(2-s)-1)$$

$$= s(2-s) \qquad s\in[0,2].$$

$$\bar{C}(x(s), y(s)) = s(2-s) \qquad s\in[0,2].$$

So we need to optimise a function of one variable on $[0,2]$

So we proceed:

1) ▶ stationary points inside $[0,2]$.

$$\frac{d}{ds}\left(s(2-s)\right) = \frac{d}{ds}\left(2s-s^2\right) = 2-2s = 0$$

$$\Longleftrightarrow s = 1$$

▶ Boundary points $s=0$, $s=2$.

2) We evaluate at these points

$$C(s,t(s))\Big|_{s=0} = 0 \qquad C(s,t(s))\Big|_{s=1} = 1\cdot(2-1) = 1$$

$$C(s,t(s))\Big|_{s=2} = 0$$

The maximum value of C along the side L_2 is 1 and is attained at the point $(1,t(1)) = (1,1)$

The minimum value of C along the side L_2 is 0 and it is attained at $(0,2)$ and $(2,0)$

▶ $C(1,1) = 1$
$C(0,t) = 0$
$C(s,0) = 0$

$$C\left(\tfrac{1}{3},\tfrac{1}{3}\right) = \frac{1}{3}\cdot\frac{1}{3}\left(\frac{1}{3}+\frac{1}{3}-1\right) = \frac{1}{9}\left(\frac{2-3}{3}\right) = -\frac{1}{27}$$

Answer

▷ The minimum value of C on the domain D is $-\frac{1}{27}$ and is attained at $(1/3, 1/3)$

▷ The maximum value of C on D is 1 and is attained at the point $(1, 1)$

Optimising functions when (x,y) are restricted to a constraint

$$g(x,y) = c$$

Example: In the example above we have optimise

$C(s,t)$ constrained to $s+t=2$.

In this case we could write $t=2-s$ (t as a function of s) but can we attack this problem in general, when we have a level curve

$$g(x,y) = c$$

We define the Lagrangian

$$\mathcal{L}(x,y) = f(x,y) + \lambda(g(x,y) - c)$$

Note

For all (x,y) such that $g(x,y) = c$, $\mathcal{L}(x,y) = f(x,y)$

Problem of optimising $f(x,y)$ subjected to $g(x,y)=c$

1) Write the Lagrangian

2) Solve $\begin{cases} \mathcal{L}'_x(x,y)=0 \\ \mathcal{L}'_y(x,y)=0 \\ g(x,y)=c \end{cases}$ (1) for (x,y,λ)

the triplets solving (1) are the solution candidates

Example Maximise $f(x,y)=x^2-y^2$ subject to the constraint $2x+y=5$,

$$\mathcal{L}(x,y)= x^2-y^2+\lambda(2x+y-5)$$

$$\left.\begin{aligned} \mathcal{L}'_x(x,y) &= 2x+2\lambda = 0 \\ \mathcal{L}'_y(x,y) &= -2y+\lambda = 0 \\ & 2x+y = 5 \end{aligned}\right\} \quad \left.\begin{aligned} &(1)\ 2x \qquad +2\lambda = 0 \\ &(2) \qquad -2y + \lambda = 0 \\ &(3)\ 2x+y \qquad = 5 \end{aligned}\right\}$$

3 equations with 3 unknowns

From (1) & (2) $\Rightarrow$ $\left[x = -\lambda \right]$ and $\left[y = \frac{1}{2}\lambda \right]$

then (3) yields $2(-\lambda) + (\frac{1}{2}\lambda) = 5 \iff (-2+\frac{1}{2})\lambda = 5$

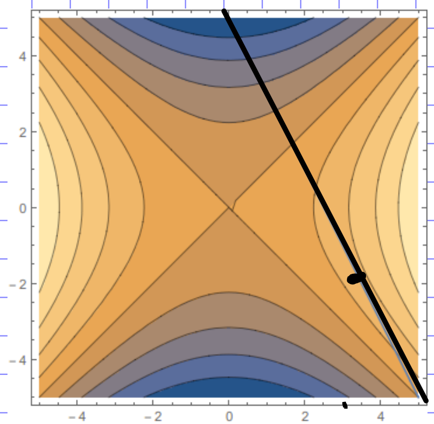
$$\iff -\frac{3}{2}\lambda = 5 \iff \boxed{\lambda = -\frac{10}{3}}$$

Hence

$$\left[\begin{aligned} x &= -\left(-\frac{10}{3}\right) = \frac{10}{3} \\ y &= \frac{1}{2}\left(-\frac{10}{3}\right) = -\frac{5}{3} \end{aligned} \right]$$

$(\frac{10}{3}, -\frac{5}{3})$ is a solution candidate

$$f\left(\frac{10}{3}, -\frac{5}{3}\right) = \frac{10^2}{9} - \frac{(-5)^2}{9} = \frac{5^2(4-1)}{9} = \frac{25}{3}.$$



$(10/3, -5/3)$

$2x+y=5$