Course MM1005

Lecture 11: Optimisation of functions of two variables

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Questions?

Lecture Goal and Outcome

Goals:

- Use the second-order partial derivatives to classify the stationary points of a function of two variables.
- Solve constrained optimisation problems with the method of substituiton and Lagrange multipliers

Learning Outcome: At the end of the lecture you will be able to solve problems like the following:

Constrained Problem

A monopolistic producer of two goods, G_1 and G_2, has a joint cost function $C = 10Q_2 + Q_2Q_1 + 10Q_1$ where Q_1 and Q_2 denote the quantities of G_1 and G_2 respectively. If P_1 and P_2 denote the corresponding prices then the demand equations are
$P_1 = 50 - Q_2 + Q_1$, $P_2 = 30 + 2Q_2 - Q_1$.
Find the maximum profit if the firm is contracted to produce a total of at most 16 goods of either type.

Unconstrained problem

A firm is a perfectly competitive producer and sells two goods G_1 and G_2 at \$800 and \$1000, respectively. The total cost of producing these goods is given by $C = 2Q_1^2 + 2Q_1Q_2 + Q_2^2$ where Q_1 and Q_2 denote the output levels of G_1 and G_2, respectively. Find the maximum profit and the values of Q_1 and Q_2 at which this is achieved.

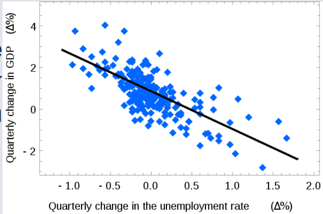
Why you should care

Linear regression

Given some data (x_t, y_y) $t = 1, \ldots, T$, we want to find a line (so $\alpha, \beta \in \mathbb{R}$) $y = \alpha + \beta x$ such that, if $e_t := \alpha + \beta x_t - y_t$ (error term at time t), it minimises the function

$$L(\alpha, \beta) = \frac{1}{T} \sum_{t=1}^{T} e_t^2$$

Does it exists? How can we find it?



Okun's Law in Macroeconomics: GDP growth is in a linear relationship with the changes in the unemployment rate

Lecture Plan

- Hessian of a function.
- Classification of stationary points.
- Sufficient conditions for extrema
- Optimisation with constraints: Substitution and Lagrange multipliers.

Recall: Necessary condition

If a function $f : D \to \mathbb{R}$ has a local extrema at a point (x_0, y_0) that is an interior point of D then

$$f'_x(x_0, y_0) = 0 \quad \text{and} \quad f'_y(x_0, y_0) = 0$$

That is (x_0, y_0) is a stationary point.

Section 1
Classification of Stationary points

The Hessian

Given a function $f : D \to \mathbb{R}$ we define its Hessian as the table (matrix)

$$H(x,y) := \begin{bmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{bmatrix}$$

and we define its determinant as

$$\det H(x,y) = f''_{xx}(x,y)f''_{yy}(x,y) - f''_{xy}(x,y)f''_{yx}(x,y)$$

For regular functions one has that $f''_{xy} = f''_{yx}$, so we could write

$$\det H(x,y) = f''_{xx}(x,y)f''_{yy}(x,y) - (f''_{xy}(x,y))^2$$

Examples: Calculate the Hessian and its determinant for $f(x,y) = -(x^2+y^2)$, $f(x,y) = x^2+y^2$ and $f(x,y) = x^2-y^2$.

Second derivative test

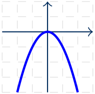
Second derivative test

Let (x_0, y_0) be a stationary point of f. Recall that

$$\det H(x_0, y_0) = f''_{xx}(x_0, y_0) f''_{yy}(x_0, y_0) - f''_{xy}(x_0, y_0) f''_{yx}(x_0, y_0)$$

1. If $\det H(x_0, y_0) > 0$ and $f''_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
2. If $\det H(x_0, y_0) > 0$ and $f''_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
3. If $\det H(x_0, y_0) < 0$ then f has a saddle point at (x_0, y_0).

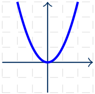
$f(x) = -x^2$



$f''(0) = -2 < 0$

Local Maximum

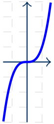
$f(x) = x^2$



$f''(0) = 2 > 0$

Local Minimum

$f(x) = x^3$



$f''(0) = 0$

$f''(x) < 0$ if $x < 0$

$f''(x) > 0$ if $x > 0$

$f'(0) = 0$

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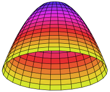
$-x^2 - y^2$

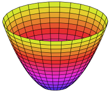


$$H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\det H = 4 > 0$

Local Maximum

$x^2 + y^2$

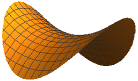


$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\det H = 4 > 0$

Local Minimum

$x^2 - y^2$



$$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\det H = -4 < 0$

Saddle Point

$f'_x(0,0) = 0$

$f'_y(0,0) = 0$

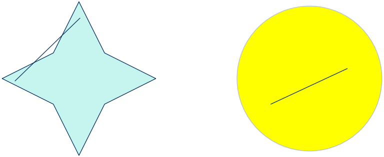
Example

(2015) Let $F(x, y) = x^4 + y^4 - 36xy$. Find all the stationary points for this function and determine whether they are local maximum, minimum or saddle points.

Section 2
Sufficient conditions for extrema

Convexity

We say that a set $D \subset \mathbb{R}^2$ is convex, if **for every** pair of points $X, Y \in D$ the segment $[X, Y]$ joining them in contained in D.



Not convex Convex

A function is *convex/concave* if the region above/below its graph is a convex set.

Sufficient conditions

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Second derivative test

Assume that:

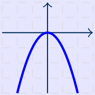
- D is a convex domain;
- (x_0, y_0) is an interior stationary point of f
- and that **for all** $(x, y) \in D$

$$\det H(x, y) = f''_{xx}(x, y) f''_{yy}(x, y) - f''_{xy}(x, y) f''_{yx}(x, y) \geq 0$$

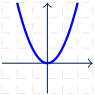
1. If $f''_{xx}(x, y) \leq 0$ and $f''_{yy}(x, y) \leq 0$ **for all** $(x, y) \in D$, then f is concave on D and (x_0, y_0) is a maximum point for f in D.
2. If $f''_{xx}(x, y) \geq 0$ and $f''_{yy}(x, y) \leq 0$ **for all** $(x, y) \in D$, then f is convex on D and (x_0, y_0) is a minimum point for f in D.

How can we recall the theorem?

$f(x) = -x^2$



$f(x) = x^2$

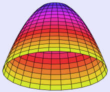


$x = 0$ is a stationary point: $f'(0) = 0$

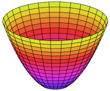
$f''(x) = -2 < 0$ (Concave)
Maximum

$f''(x) = 2 > 0$ (Convex)
Minimum

$f(x, y) = -(x^2 + y^2)$



$f(x, y) = x^2 + y^2$



$(0, 0)$ is a stationary point $f'_x(0, 0) = f'_y(0, 0) = 0$

$$H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\det H = f''_{xx} f''_{yy} - (f''_{xy})^2 \geq 0$
$f''_{xx}(x, y) \leq 0 \quad f''_{yy}(x, y) \leq 0$
(Concave) Maximum

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\det H = f''_{xx} f''_{yy} - (f''_{xy})^2 > 0$
$f''_{xx}(x, y) \geq 0 \quad f''_{yy}(x, y) \geq 0$
(Convex) Minimum

Example

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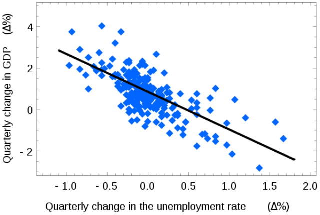
Show that the function

$$f(x, y) = 2x^2 + 4xy + 3y^2 - x - 2y$$

is convex and has a global minimum on $\mathbb{R}^2$.

Application:Linear regression

Given some data (x_t, y_y) $t = 1, \ldots, T$, we want to find a line (so $\alpha, \beta \in \mathbb{R}$) $y = \alpha + \beta x$ such that, if $e_t := \alpha + \beta x_t - y_t$ (error term at time t), it minimises the function



$$L(\alpha, \beta) = \frac{1}{T} \sum_{t=1}^{T} e_t^2$$

Define $\mu_x = \frac{1}{T} \sum_{t=1}^{T} x_t \quad \mu_y = \frac{1}{T} \sum_{t=1}^{T} y_t$ and

$$\sigma_{xx} = \frac{1}{T} \sum_{t=1}^{T} (x_t - \mu_x)^2, \qquad \sigma_{yy} = \frac{1}{T} \sum_{t=1}^{T} (y_t - \mu_y)^2,$$

$$\sigma_{xy} = \frac{1}{T} \sum_{t=1}^{T} (x_t - \mu_x)(y_t - \mu_y)$$

Assume that $\sigma_{xx} > 0$ (otherwise, all the points lie on a vertical line)

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One can show (afer some calculations) that $L(\alpha, \beta) =$
$= \alpha^2 + \mu_y^2 + \beta^2\mu_x^2 - 2\alpha\mu_y - 2\beta\mu_x\mu_y + 2\alpha\beta\mu_x + \beta^2\sigma_{xx} - 2\beta\sigma_{xy} + \sigma_{yy}$
Also that the unique stationary point is

$$\hat{\alpha} = \mu_y - \frac{\sigma_{xy}}{\sigma_{xx}}\mu_x, \qquad \hat{\beta} = \frac{\sigma_{xy}}{\sigma_{xx}}$$

Note that

$$H(\alpha, \beta) = \begin{bmatrix} 2 & 2\mu_x \\ 2\mu_x & 2\mu_x^2 + 2\sigma_{xx} \end{bmatrix} \Rightarrow \det H(\alpha, \beta) = \sigma_{xx} > 0$$
$$2 = H''_{\alpha\alpha} > 0 \qquad 2\mu_x^2 + 2\sigma_{xx} = H''_{\beta\beta} > 0$$

Since $\mathbb{R}^2$ is convex, by the previous theorem, L is convex and the stationary point is a global minimum for $L(\alpha, \beta)$.

$y = \hat{\alpha} + \hat{\beta}x$ is called the regression line.

Section 3
Constrained Optimisation

The EVT

1D case

Let $f : [a, b] \to \mathbb{R}$ differentiable
$[a, b]$ is closed and bounded

1. There exists $x_m, x_M \in [a, b]$ such that for all $x \in [a, b]$

$$f(x_m) \leq f(x) \leq f(x_M)$$

2. The Maximum/minimun points are either
    1. A stationary point inside $[a, b]$; or
    2. A point in the border (either a or b).

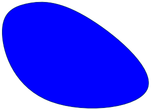
2D case

Let $f : D \to \mathbb{R}$ differentiable
D is closed and bounded

1. There exists $x_m, x_M \in D$ such that for all $x \in D$

$$f(x_m) \leq f(x) \leq f(x_M)$$

2. The Maximum/minimun points are either
    1. A stationary point inside D; or
    2. A point in the border (is a curve).



Optimisation problem

The strategy is essentially the same one as for functions in one variable.
Given $f : D \to \mathbb{R}$, where D is closed and bounded.

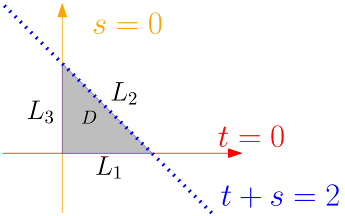
1. Make a list consisting of
   - Stationary points inside D;
   - Extremal "Boundary points"
2. Evaluate f at all the points in the list
3. Find the largest/smallest value

To find the extremes in the boundary we need to describe it/parametrise it.

Example

(2018) A quantity C is determined by inputs s, t, according to the formula $C(s,t) = st(s+t-1)$. Find the maximum and minimum possible values for C subjected to the constraints $s \geq 0$, $t \geq 0$ and $s + t \leq 2$.

Lets start by visualising the domain D:



The border of D consist of three pieces:

1. L_1 is the set of points (s,t) such that $t = 0$, $s \in [0,2]$;
2. L_3 is the set of points (s,t) such that $s = 0$, $t \in [0,2]$;
3. L_2 is the set of points (s,t) such that $t = 2 - s$, $s \in [0,2]$.

Stationary points inside D

Recall that $C(s,t) = st(s+t-1)$
The partial derivative are:

$$\begin{cases} C_s'(s,t) = t(2s+t-1) = 0 \Leftrightarrow t = 0 \text{ or } 2s+t-1=0 \\ C_t'(s,t) = s(2t+s-1) = 0 \Leftrightarrow s = 0 \text{ or } 2t+s-1=0 \end{cases}$$

So the stationary points are those (s,t) satisfying:

1. $t = 0 = s$ i.e. $(0,0)$.
2. $t = 0$ and $2t+s-1=0$ i.e. $(1,0)$;
3. $s = 0$ and $2s+t-1=0$ i.e. $(0,1)$;
4. $2s+t-1=0$ and $2t+s-1=0$ i.e. $(1/3,1/3)$;

Only $(s,t) = (1/3,1/3)$ lies inside D.

On the Border

Recall that $\qquad C(s,t) = st(s+t-1)$

We look for the maximum/minimum values of the function when it is restricted to the border (i.e. $C(s,t)$ when (s,t) belong to the border):

- Alongside L_1 ($t = 0$, $s \in [0,2]$) $C(s,t) = 0$. The function is constant on L_1;
- Alongside L_3 ($s = 0$, $t \in [0,2]$) $C(s,t) = 0$. The function is constant on L_3;
- Alongside L_2 ($t = 2 - s$ and $s \in [0,2]$)

  $$C(s,t) = C(s,2-s) = s(2-s)(2+(2-s)-1) = s(2-s)$$

  So we need to optimise a function of one variable on $s \in [0,2]$.

$$C(s,t) = s(2-s), \qquad s = s,\ t = 2-s,\ s \in [0,2],$$

We proceed by

1. Making a list of:
   - Stationary points inside $[0,2]$:
     $$\frac{\mathrm{d}}{\mathrm{d}s}(s(2-s)) = 2 - 2s = 0 \Leftrightarrow s = 1;$$
   - (Extremal) Boundary points: $s = 0$, $s = 2$;
2. Evaluate the function in those points
   - For $s = 0$ (so $t = 2$): $C(0,2) = 0$;
   - For $s = 2$ (so $t = 0$): $C(2,0) = 0$;
   - For $s = 1$ (so $t = 1$): $C(1,1) = 1(2-1) = 1$

The maximum value of C on L_2 is 1 and it is attained at $(1,1)$;
The minimum value of C on L_2 is 0 and it is attained at $(0,2)$ & $(2,0)$.

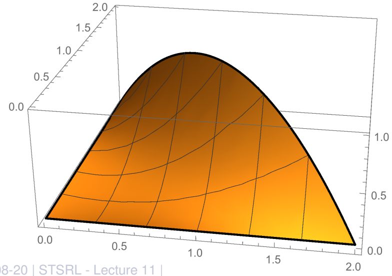
Evaluation and comparison

The list of points we have are

- $C(1,1) = 1$;
- $C(0,t) = 0$, $t \in [0,2]$;
- $C(s,0) = 0$, $s \in [0,2]$;
- $C(1/3, 1/3) = \frac{1}{3}\frac{1}{3}(\frac{1}{3} + \frac{1}{3} - 1) = -\frac{1}{27}$

As a result of the analysis above we have that:

- The maximum value of C on the domain D is 1 and it is attained at the points $(1,1)$;
- The minimum value of C on the domain D is $-1/27$ and it is attained at $(1/3, 1/3)$.



Lagrange Multipliers

In some cases as above, we need to solve an optimisation problem $C(x, y)$ under some constrains of the form

$$g(x, y) = 0,$$

Example: In the previous example, we had to optimise $C(s, t)$ constrained to $s + t - 2 = 0$.

In that case, it was easy to write t as a function of s ($t = 2 - s$), plug that in the function $C(s, t)$ and treat that as a function in one variable. This is called the **substitution method**.

In general, one can't explicitly write one of the variables as a function of the other, but one can try by the **method of Lagrange Multipliers**

Lagrange multipliers method

Optimise the function $f(x, y)$ with the constraint $g(x, y) = 0$.

1. Construct the Lagrangian function

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y),$$

2. The candidates for relative extrema occur at the solutions of

$$\begin{cases} \mathcal{L}'_x(x, y) = 0 \\ \mathcal{L}'_y(x, y) = 0 \\ g(x, y) = 0 \end{cases}$$

Be careful! The method only produce candidates for relative extrema. In every problem we'll need to make sure that those extrema exist.

Example

Maximise $f(x,y) = x^2 - y^2$ subject to the constraint $2x + y - 5 = 0$.
Lagrange method: We construct the Lagangian:

$$\mathcal{L}(x,y) = x^2 - y^2 + \lambda(2x + y - 5),$$

We (try to) solve the system of 3 equations an 3-unknowns (x, y, λ):

$$\begin{cases} 0 = \mathcal{L}'_x(x,y) = 2x + 2\lambda \\ 0 = \mathcal{L}'_y(x,y) = -2y + \lambda \\ 0 = 2x + y - 5 \end{cases} \Leftrightarrow \begin{cases} 0 = 2x \qquad + 2\lambda \\ 0 = \qquad - 2y + \lambda \\ 5 = 2x + \quad y \end{cases}$$

From the first and second rows we have $x = -\lambda, \qquad y = \lambda/2,$
Plugging this in the third one yields

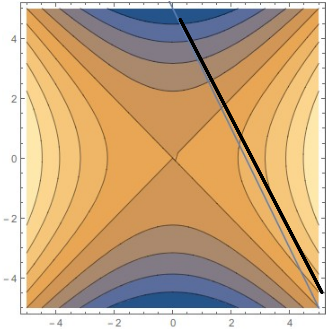
$$5 = 2(-\lambda) + (\lambda/2) = -3\lambda/2 \Leftrightarrow \lambda = -10/3$$

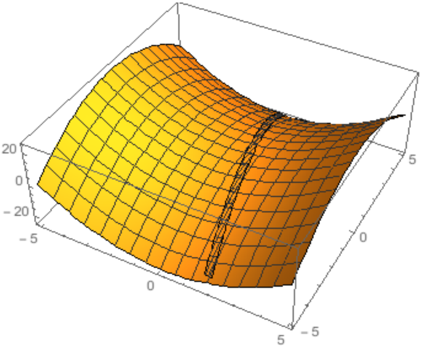
Hence

$$x = 10/3,\ y = -5/3 \quad f(10/3, -5/3) = (100 - 25)/9 = 25/3$$

Graphically

Looking at the level curves and the graph of f, we can see that the function, constrained to the line, has a global maximum at that point.





Thank you for your attention!