

Course MM1005

Lecture 8: Geometric Series

Sofia Tirabassi

tirabassi@math.su.se

Salvador Rodriguez Lopez

s.rodriquez-lopez@math.su.se

Questions?

Lecture Goal and Outcome

Goals:

- Recognise a geometric progression.
- Evaluate a geometric series.
- Determine if a geometric series is convergent or not.
- Use these concepts in some applications.

Learning Outcome: At the end of the lecture you will be able to solve problem like the following:

Problem

Determine for which real values of x the geometric series $3 + 6/x + 12/x^2 + 24/x^3 + \dots$ converges, and for which value of x the series is equal to 5.

Why you should care

Retirement value

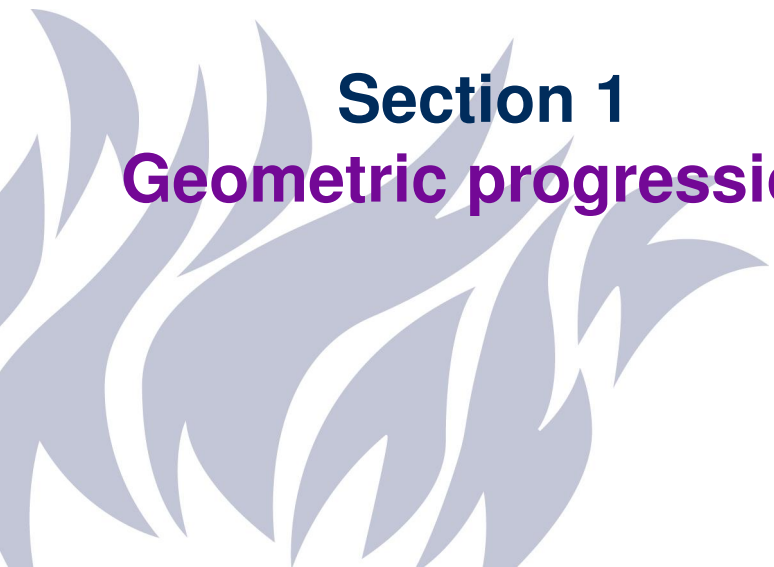
Your retirement account has \$5000 in it and earns 5% interest per year compounded monthly. Every month for the next 10 years you will deposit \$100 into the account. How much money will there be in the account at the end of those 10 years?

Saving account values

Kalle and Maria have just had a daughter, and have decided to open a bank account to save money for her university education. They would like to have \$100000 after 17 years. If the account pays 4% interest per year, compounded quarterly, and they make deposits at the end of every quarter, how large must each deposit be for them to reach their goal?

Lecture Plan

- §11.4 Geometric Series
- (§11.5 Total present value)
- (§11.6 Mortgage repayments)



Section 1

Geometric progression

Geometric progression

If we invest \$100 at 10% interest compounded annually, then the future values in successive years are:

$$\begin{aligned}a_0 &= 100, & a_1 &= 100(1,1), & a_2 &= 100(1,1)^2, \\a_3 &= 100(1,1)^3, \dots, & a_n &= 100(1,1)^n\end{aligned}$$

In general, if we invest \$P at I% interest compounded annually, then the future values in successive years are:

$$\begin{aligned}a_0 &= P, & a_1 &= P(1 + \frac{I}{100}), & a_2 &= P(1 + \frac{I}{100})^2, \\a_3 &= P(1 + \frac{I}{100})^3, \dots, & a_n &= P(1 + \frac{I}{100})^n\end{aligned}$$

Note that

$$1 + \frac{I}{100} = \frac{a_1}{a_0} = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \dots = \frac{a_{n+1}}{a_n}$$

Definition

We say that a (possibly infinite) sequence of numbers $a_0, a_1, a_2, \dots, a_n, \dots$ is a **geometric progression** if there exists a constant r (called the **(geometric) ratio**) such that for all $n \geq 1$

$$r = \frac{a_{n+1}}{a_n}$$

In this case we can write $a_n = a_0 r^n$, $n \geq 0$,

Example The formula $a_n = P(1 + \frac{I}{100})^n$ defines a geometric sequence with ratio $(1 + \frac{I}{100})$.

Example Which of the following sequences are geometric progressions? For those sequences that are of this type, write down their geometric ratios:

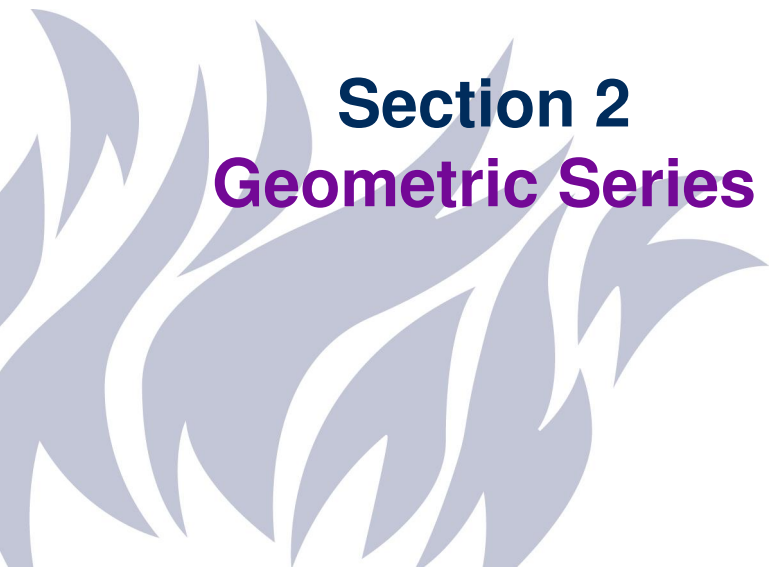
a) 1000, -100, 10, -1, ...

b) 3, 6, 9, 12, ...

c) 3, 6, 12, 24, ...

d) -1, 3, -9, 27, ...

You have \$5000 to invest at 6%/year interest compounded monthly. How long will it take for your investment to grow to \$6000?



Section 2

Geometric Series

Geometric Series

If you make a payment of \$100 at the end of every month into an account earning 3.6%/year, compounded monthly (this means that your investment is earning $3.6\% / 12 = 0.3\%$ /month). What will be the value of the investment at the end of 1 year (12 months)?

The first payment will be compounded 11 times. So it will produce $100(1,003)^{11}$ at the end of the period.

The second payment will be compounded 10 times. It will produce $100(1,003)^{10}$ at the end of the period.

⋮

The n th payment will be compounded $12 - n$ times. It will produce $100(1,003)^{12-n}$ at the end of the period.

1	2	3	4	5	6	7	8	9	10	11	12
p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$	$p \cdot r^7$	$p \cdot r^8$	$p \cdot r^9$	$p \cdot r^{10}$	$p \cdot r^{11}$
	p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$	$p \cdot r^7$	$p \cdot r^8$	$p \cdot r^9$	$p \cdot r^{10}$
		p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$	$p \cdot r^7$	$p \cdot r^8$	$p \cdot r^9$
			p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$	$p \cdot r^7$	$p \cdot r^8$
				p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$	$p \cdot r^7$
					p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$	$p \cdot r^6$
						p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$	$p \cdot r^5$
							p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$	$p \cdot r^4$
								p	$p \cdot r$	$p \cdot r^2$	$p \cdot r^3$
									p	$p \cdot r$	$p \cdot r^2$
										p	$p \cdot r$
											p

The first payment will be compounded 11 times. So it will produce $100(1,003)^{11}$ at the end of the period.

The second payment will be compounded 10 times. So it will produce $100(1,003)^{11}$ at the end of the period.

The 12th payment won't be compounded.

The value of the investment at the end of 12 months is

$$pr^0 + pr^1 + pr^2 + \dots + pr^{11}$$

Geometric Series

Given a geometric progression

$$a_n = a_0 r^n$$

we are interested in calculate the finite **geometric sums**

$$s_n := \sum_{k=0}^n a_0 r^k = a_0(r^0 + r^1 + \dots + r^n), \quad n \geq 0$$

Note that

$$\begin{aligned} r s_n &= a_0(r^1 + r^2 + \dots + r^{n+1}) \\ &= a_0(r^0 + r^1 + r^2 + \dots + r^n + r^{n+1} - r^0) = s_n - a_0(1 - r^{n+1}), \end{aligned}$$

This implies that (if $r \neq 1$)

$$a_0(1 - r^{n+1}) = (1 - r)s_n \Leftrightarrow s_n = a_0 \frac{1 - r^{n+1}}{1 - r}$$

Finite Geometric Series

$$\sum_{k=0}^n a_0 r^k = \begin{cases} a_0 \frac{1-r^{n+1}}{1-r} & \text{if } r \neq 1 \\ a_0(n+1) & \text{if } r = 1 \end{cases} \quad n \geq 0$$

Example In the previous example, we reduce the problem to calculate, for $p = 100$, $r = 1,003$, $n = 11$

$$\begin{aligned} pr^0 + pr^1 + pr^2 + \dots + pr^{11} &= p \frac{1-r^{12}}{1-r} = 100 \frac{1-(1,003)^{12}}{-0,003} \\ &\approx 1220 \end{aligned}$$

Example

If you make a payment of \$100 at the end of every month into an account earning 3.6%/year, compounded monthly. This means that your investment is earning $3.6\% / 12 = 0.3\%$ /month. How many months will have to pass for the value of the investment to be at least \$100000?

Now, what we don't know is the number of months, say n . We would like to find n such that

$$pr^0 + pr^1 + pr^2 + \dots + pr^{n-1} = 100 \frac{(1,003)^n - 1}{0,003} \geq 100000$$

which is equivalent to find a natural number n such that

$$(1,003)^n \geq 3 + 1 \Leftrightarrow n \geq \frac{\ln(4)}{\ln 1,003} \approx 462,7$$

So we need to wait at least 463 months that is ≈ 38 years!

If n successive annual payments from a deposit are to be made, with the same amount $\$a$ (due after every year). How much must be deposited in the account today in order to have enough savings to cover these n payments, given that the interest rate is $l\%/year$?

In order to have the amount after

- one year, we must deposit an amount x_1

$$x_1(1+l) = a \Leftrightarrow x_n = a(1+l)^{-1}$$

- two year, we must deposit an amount x_2

$$x_2(1+l)^2 = a \Leftrightarrow x_n = a(1+l)^{-2}$$

- n years, we must deposit an amount x_n

$$x_n(1+l)^n = a \Leftrightarrow x_n = a(1+l)^{-n}$$

So the total amount deposit today has to be $a \sum_{j=1}^n ((1+l)^{-1})^j$

N.b. An **annuity** is a sequence of equal payments made at fixed periods of time over some time span.

Geometric series

$$\sum_{k=l}^n a_0 r^k = r^l \sum_{k=0}^{n-l} a_0 r^k = \begin{cases} a_0 r^l \frac{1-r^{n-l+1}}{1-r} & \text{if } r \neq 1 \\ a_0 (n-l+1) & \text{if } r = 1 \end{cases}$$

In the example above

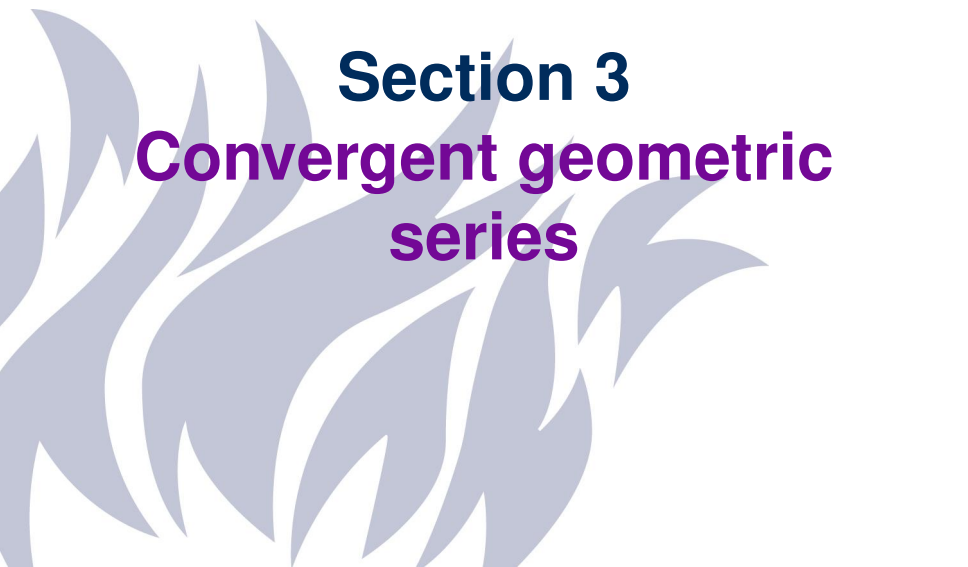
$$\begin{aligned} a \sum_{j=1}^n ((1+l)^{-1})^j &= a(1+l)^{-1} \sum_{j=0}^{n-1} ((1+l)^{-1})^j \\ &= \frac{a}{l} \left(1 - \frac{1}{(1+l)^n} \right) \end{aligned}$$

This formula gives the present value of n future claims of \$ a /each.

In the same situation as above, how much be deposited in the account today in order to cover an **infinite** number of payments?
To find the answer, we need to calculate

$$\begin{aligned}\lim_{n \rightarrow \infty} a \sum_{j=1}^n ((1 + I)^{-1})^j &= \lim_{n \rightarrow \infty} \frac{a}{I} \left(1 - \frac{1}{(1 + I)^n} \right) \\ &= \frac{a}{I} \left(1 - \lim_{n \rightarrow \infty} \frac{1}{(1 + I)^n} \right) \\ &= \frac{a}{I},\end{aligned}$$

Depositing $\frac{a}{I}$ we make sure that we can withdraw an amount \$a/year forever.



Section 3

Convergent geometric series

Convergent Series

We say that the geometric series $\sum_{j=l}^n r^k$ **converges** if $\lim_{n \rightarrow \infty} \sum_{j=l}^n r^k$ exists as a real number. If so, we denote

$$\sum_{k=l}^{\infty} r^k := \lim_{n \rightarrow \infty} \sum_{j=l}^n r^k$$

Notice that for $r \neq 1$

$$\lim_{n \rightarrow \infty} \sum_{k=l}^n r^k = \lim_{n \rightarrow \infty} r^l \frac{1 - r^{n-l+1}}{1 - r} = \begin{cases} \frac{r^l}{1-r} & \text{if } |r| < 1 \\ \infty & \text{if } |r| > 1 \end{cases}$$

Convergent Series

The geometric series $\sum_{j=l}^n r^k$ converges if and only if $|r| < 1$.

Examples

(2008)

- 1 For which real numbers is the series $S = 1 + \frac{3}{5}e^x + \frac{9}{25}e^{2x} + \frac{27}{125}e^{3x} + \dots$ convergent?
- 2 Find x such that $S = 5$

(2016) Find the values of a for which the series $\sum_{k \geq 0} \left(\frac{1}{a^2 - 1} \right)^k$ converges.

(2021) Determine for which x is the series $\sum_{n=0}^{\infty} (1+x)^{-2n}$ convergent, and when does the sum equals 2.



Section 4

Applications

Applications

You wish to establish a trust fund from which your daughter can withdraw \$1,000 every six months for 10 years. At the end of that time, she will get the remaining money in the trust, which you would like to be \$50,000. The trust will be invested at 7%/year compounded every six months. How large should the trust be?

A person borrows \$50 000 at the beginning of a year and is supposed to pay it off in 5 equal instalments at the end of each year, with interest at 15% compounding annually. Find the annual payment.

If \$a is amount of each repayment, the present value is

$$\begin{aligned}\sum_{k=1}^5 \frac{a}{(1,15)^k} &= a(1,15)^{-1} \sum_{k=0}^4 \left(\frac{1}{1,15} \right)^k \\ &= a(1,15)^{-1} \frac{1 - (1,15)^{-5}}{1 - (1,15)^{-1}} = \frac{a}{0,15} (1 - (1,15)^{-5})\end{aligned}$$

This quantity has to be equal to 50 000, so

$$\frac{a}{0,15} (1 - (1,15)^{-5}) = 50000 \Leftrightarrow a \approx 14915,78$$

Suppose the loan of \$50 000 is to be repaid by paying \$20 000, which covers both interest and the principal repayment, at the end of each of the coming years, until the loan is fully paid off. When is the loan paid off if the annual rate is 15%?

In this case we need to find n such that

$$\sum_{k=1}^n \frac{2 \times 10^4}{(1,15)^k} = \frac{2 \times 10^4}{0,15} (1 - (1,15)^{-n}) \geq 5 \times 10^4,$$

This is equivalent to solve

$$\begin{aligned} 1 - (1,15)^{-n} &\geq \frac{5 \times 15}{2 \times 100} \Leftrightarrow \ln(1 - \frac{3}{8}) \geq -n \ln(1,15) \\ \Leftrightarrow n &\geq \frac{\ln(\frac{8}{5})}{\ln(1,15)} \approx 3,36 \end{aligned}$$

Thank you for your attention!

