# Course MM1005

## Lecture 13: Matrices and Linear Systems-I

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# Questions?

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# Lecture Goal and Outcome

**Goals:**

- Introduce the notion of vectors and matrices, their operations, and the connection to linear systems.
- Introduce the Gauss elimination method to solve linear systems.

**Learning Outcome:** At the end of the lecture you will be able to solve problem like the following:

**Problem**

The demand and supply equations of a good are given by

$$4P = -Q_d + 240, \qquad 5P = Q_s + 30,$$

where $P$ is the price of the good, $Q_d$ is the quantity demanded, and $Q_s$ is the quantity supplied to the market.
Find the values of $P, Q_s, Q_f$ such that the market is in equilibrium.

# Why you should care

## The Portfolio problem

Given a wealth $W$, we want to invest it in a collection of $n$ risky assets $\{S_1, \ldots, S_n\}$: a fraction of our wealth, $w_i$ say, in asset $S_i$. Assuming that short selling is allowed (some of the $w_i$ may be negative (borrow)).

> *How can we choose an optimal set of weights $\{w_1, ,,,, w_n\}$, so that our overall investment is likely to yield a promising return with minimal risk?*

To attack this problem, you need to learn several mathematical tools: Linear Algebra, Optimisation and Probability.

# Lecture Plan

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- Vectors & Matrices
- Matrix operations
- System of linear equations & Gaussian Elimination

# Section 1
# Vectors and Matrices

# Vectors

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We can represent a portfolio of *n*-stocks (or other assets) with a table of *n* rows and 1 column (vector) as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

where $x_i$ shares of stock $i = 1 \ldots, m$. We can define its transpose as

$$\vec{x}^T = \begin{bmatrix} x_1 & x_2 & \ldots & x_m \end{bmatrix}$$

Similarly, we can write

$$\vec{s}^T = \begin{bmatrix} s_1 & s_2 & \dots & s_n \end{bmatrix}$$

where $s_i$ represents the price of a share of asset $i$ under a particular economic scenario (or the )
The overall value of the portfolio under the given scenario is

$$s_1 x_1 + s_2 x_2 + \ldots + s_n x_n$$

Multiplication of a $(1 \times n)$ row vector and and $(n \times 1)$ column vector

$$\vec{s}^T \vec{x} := \begin{bmatrix} s_1 & s_2 & \dots & s_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = s_1 x_1 + s_2 x_2 + \ldots + s_n x_n$$

# Examples

- $\vec{s}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\vec{x}^T = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}$

Suppose now that we have $m$ different scenarios, each with different price for share $i$:

$$\vec{s_j}^T = \begin{bmatrix} s_{j1} & s_{j2} & \dots & s_{jn} \end{bmatrix} \qquad j = 1, \dots, m,$$

We can represent all of them in a rectangular array of numbers with $m$ rows and $n$ columns ($m \times n$-Matrix):

$$S = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} = \begin{bmatrix} \vec{s_1}^T \\ \vec{s_2}^T \\ \vdots \\ \vec{s_m}^T \end{bmatrix}$$

The overall value of the portfolio for the $m$ scenarios is given by the $m \times 1$-matrix

$$S\vec{x} := \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \vec{s_1}^T \vec{x} \\ \vec{s_2}^T \vec{x} \\ \vdots \\ \vec{s_m}^T \vec{x} \end{bmatrix}$$

**Examples**

$$\begin{bmatrix} 3 & -7 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 7 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix},$$

Suppose now, that we want to manage $k$-different portfolios, which we could represent now as a $n \times k$-Matrix

$$X := \left[\vec{x}_1 \quad \vec{x}_2 \ldots \quad \vec{x}_k\right] = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1k} \\ x_{21} & x_{22} & \ldots & x_{2k} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nk} \end{bmatrix}$$

The overall value of the $k$ different portfolios under $m$ different scenarios is given by the $m \times K$ matrix

$$S{\cdot}X = \left[Sx_1 \quad \ldots Sx_k\right] = \begin{bmatrix} s_{11} & s_{12} & \ldots & s_{1n} \\ s_{21} & s_{22} & \ldots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \ldots & s_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1k} \\ x_{21} & x_{22} & \ldots & x_{2k} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nk} \end{bmatrix}$$

Multiplication of a $(m \times n)$ Matrix $S$ and and $(n \times k)$ Matrix $X$

$$\begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nk} \end{bmatrix}$$

gives a $m \times k$-matrix $C$ where the element in the row $i$ and column $j$ is given by the matrix multiplication of the row $i$ of $S$ and the $j$ column of $X$

$$c_{ij} = \begin{bmatrix} s_{i1} & s_{i2} & \dots & s_{in} \end{bmatrix} \cdot \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{bmatrix} = s_{i1}x_{1j} + s_{i1}x_{2j} + \dots + s_{in}x_{nk}$$

# Examples

$$\begin{bmatrix} 3 & -7 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 8 & 4 \\ 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix},$$

# Comparison of two matrices

Two matrices $(a_{ij})_{m\times n}$, $(b_{ij})_{k\times l}$ are equal if

1. Have the same order (same number of rows and columns) (i.e. $m = k$, $n = l$)
2. For all $i = 1, \dots, m$ and $j = 1, \dots, n$

$$a_{ij} = b_{ij},$$

**Examples**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

For which values of $x_1, x_2, x_3$ are the two following matrices equal?

$$\begin{bmatrix} 4 & x_2 \\ x_3 & 4 \end{bmatrix} = \begin{bmatrix} x_1 & 3 \\ 5 & x_4 \end{bmatrix}$$

# Sum of matrices

Whenever we have to matrices of the same order (same number of rows and columns) we can add them together term by term:

Given $S, X$ two $m \times n$ matrices, we define $S + X$ as

$$\begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} + \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mk} \end{bmatrix}$$

$$= \begin{bmatrix} s_{11} + x_{11} & s_{12} + x_{12} & \dots & s_{1n} + x_{1n} \\ s_{21} + x_{21} & s_{22} + x_{22} & \dots & s_{2n} + x_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} + x_{m1} & s_{m2} + x_{m2} & \dots & s_{mn} + x_{mn} \end{bmatrix}$$

# Product by an scalar

Given an $m \times n$ matrix $S$, and a real value $a$ we define $aS$ as

$$\begin{bmatrix} as_{11} & aas_{12} & \dots & as_{1n} \\ as_{21} & as_{22} & \dots & as_{2n} \\ \vdots & & \ddots & \vdots \\ as_{m1} & as_{m2} & \dots & as_{mn} \end{bmatrix}$$

**Examples:**

$$3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 7 \end{bmatrix},$$

# Transpose

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If $\vec{x} = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}$ its transpose is $\vec{x}^T = \begin{bmatrix} x_{11} & x_{21} & \dots & x_{n1} \end{bmatrix}$

In general

$$X = \begin{bmatrix} \color{blue}{x_{11}} & \color{red}{x_{12}} & \dots & x_{1k} \\ \color{blue}{x_{21}} & \color{red}{x_{22}} & \dots & x_{2k} \\ \vdots & & \ddots & \vdots \\ \color{blue}{x_{n1}} & \color{red}{x_{n2}} & \dots & x_{nk} \end{bmatrix} \text{ then } X^T = \begin{bmatrix} \color{blue}{x_{11}} & \color{blue}{x_{21}} & \dots & \color{blue}{x_{n1}} \\ \color{red}{x_{12}} & \color{red}{x_{22}} & \dots & \color{red}{x_{n2}} \\ \vdots & & \ddots & \vdots \\ x_{1k} & x_{2k} & \dots & x_{nk} \end{bmatrix}$$

If $A, B$ are $m \times n$-matrices, $C$ $n \times k$-matrix, and $a \in \mathbb{R}$:

1. $(A^T)^T = A$
2. $(AC)^T = C^T A^T$
3. $(A + B)^T = A^T + B^T$
4. $(aA)^T = aA^T$

# Section 2
# Linear Systems

# Linear systems: Motivation-I

The matrix $S$ below represents the net profits to 3 stocks under 4 outcomes or scenarios:

$$S := \begin{bmatrix} -2 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 6 \\ 1 & -1 & -5 \end{bmatrix}$$

Can we invest in a way that produces the vector $\vec{v} = \begin{bmatrix} 0 & 0 & 0 & 5 \end{bmatrix}^T$?
How can we figure out the answer to this?
We look for a vector $\vec{x} = \begin{bmatrix} x & y & z \end{bmatrix}^T$ such that

$$\begin{bmatrix} -2 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 6 \\ 1 & -1 & -5 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\textit{Unknown}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \Leftrightarrow \begin{cases} -2x + 3y + z & = & 0 \\ -x + 2y & = & 0 \\ \quad - y + 6z & = & 0 \\ x - y - 5z & = & 5 \end{cases}$$

# Linear systems: Motivation-II

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The demand and supply equations of a good are given by

$$4P = -Q_d + 240, \qquad 5P = Q_s + 30,$$

where $P$ is the price of the good, $Q_d$ is the quantity demanded, and $Q_s$ is the quantity supplied to the market.
At the point of intersection of the demand and supply curves (i.e. $Q_s = Q_p =: Q$), the market is said to be in equilibrium because the quantity demanded is equal to the quantity supplied.
How can we find that equilibrium point?
We need to find $(P, Q)$ such that

$$\begin{cases} 4P + Q &= 240 \\ 5P - Q &= 30 \end{cases} \Leftrightarrow \begin{bmatrix} 4 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 240 \\ 30 \end{bmatrix}$$

# Linear systems

A linear system of $m$-(linear) equations and $n$-unknowns $x_1, \ldots x_n$

$$\left\{ \begin{array}{c} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_1 \end{array} \right. \Leftrightarrow A\vec{x} = \vec{b},$$

where

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$A$ is called the coefficient matrix $\quad \vec{x}$ unknown $\quad \vec{b}$ independent term.

Taking the unknowns for granted, we can write the system in a compact form (Augmented coefficient matrix):

$$A\vec{x} = \vec{b} \Leftrightarrow \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right]$$

**Examples**

$$\begin{bmatrix} 4 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 240 \\ 30 \end{bmatrix} \Leftrightarrow \left[\begin{array}{cc|c} 4 & 1 & 240 \\ 5 & -1 & 30 \end{array}\right]$$

$$\begin{bmatrix} -2 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 6 \\ 1 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 6 & 0 \\ 1 & -1 & -5 & 5 \end{array}\right]$$

# Simple systems $Ax = b$

The equation...

1. $1 \cdot x = 1$ has a unique solution $x = 1$.
2. The equation $0 \cdot x = 0$ is satisfied for all $x \in \mathbb{R}$. Has an infinite number of solutions.
3. $0 \cdot x = 1$ has no solution.

The system of equations...

1. $\left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 30 \end{array}\right] \Leftrightarrow \begin{cases} x = 10 \\ y = 30 \end{cases}$ has only one solution.
2. $\left[\begin{array}{cc|c} 1 & 1 & 10 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x + y = 10 \\ 0 \cdot x + 0 \cdot y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 10 - t \\ y = t \end{cases} \quad t \in \mathbb{R},$
   The system has an infinite number of solutions.
3. $\left[\begin{array}{cc|c} 1 & 1 & 10 \\ 0 & 0 & 1 \end{array}\right] \Leftrightarrow \begin{cases} x + y = 10 \\ 0 = 1 \end{cases}$ has no solution.

# Gauss elimination method

Strategy: Simplify the system by using the following elementary operations:

1. Exchange two equations (rows);
2. Multiplying equations (rows) by scalars;
3. Adding a multiple of one equation to another

**Example**

$$\left[\begin{array}{cc|c} 5 & -3 & 25 \\ 1 & 1 & 6 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 5 & -3 & 25 \end{array}\right] \underset{-5R_1+R_2 \to R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & -8 & -5 \end{array}\right]$$

$$\underset{R_2+8R_1 \to R_1}{\sim} \left[\begin{array}{cc|c} 8 & 0 & 43 \\ 0 & -8 & -5 \end{array}\right] \underset{\substack{R_1/8 \to R_1 \\ -R_2/8 \to R_2}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 43/8 \\ 0 & 1 & 5/8 \end{array}\right]$$

If we want to solve a bunch of systems with the same coefficients, but different independent terms

$$Ax = \vec{b}_1, \quad Ax = \vec{b}_2 \ldots \quad Ax = \vec{b}_k$$

we can run the method for all of them at the same time by writing

$$\left[\begin{array}{c|cccc} A & b_1 & b_2 & \ldots & b_k \end{array}\right]$$

**Example** Determine if any of following systems has any solution. If so, find all solutions.

1. $\begin{cases} 2x + y = 1 \\ 4x + 2y = 3 \end{cases}$
2. $\begin{cases} 2x + y = 1 \\ 4x + 2y = 2 \end{cases}$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 1 \\ 4 & 2 & 3 & 2 \end{array}\right] \underset{-2R_1+R_2 \to R_2}{\sim} \left[\begin{array}{cc|cc} 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

# Exercises

Determine if the systems of the motivational examples have any solution. If so, find the solution(s):

1. $\left[\begin{array}{cc|c} 4 & 1 & 240 \\ 5 & -1 & 30 \end{array}\right]$. $P = 30$, $Q = 120$
2. $\left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 6 & 0 \\ 1 & -1 & -5 & 5 \end{array}\right]$

# Multicomodity market

The demand and supply functions for two interdependent commodities are given by

$$\begin{aligned} Q_{d1} &= a_1 + b_1 P_1 + c_1 P_2 \\ Q_{d2} &= a_2 + b_2 P_1 + c_2 P_2 \\ Q_{s1} &= a_3 + b_3 P_1 \\ Q_{s2} &= a_3 + c_3 P_2 \end{aligned}$$

where $P_i$, $Q_{di}$ and $Q_{ds}$ denote the price and quantity demanded, and quantity supplied for the $i$th good, and $a_i, b_i, c_i$ are constants depending on the model.
Determine the equilibrium price and quantity for this two-commodity model (i.e. $Q_{di} = Q_{si}$)

# Thank you for your attention!