Lecture 11

1) Optimisation under a constraint: Lagrange multipliers

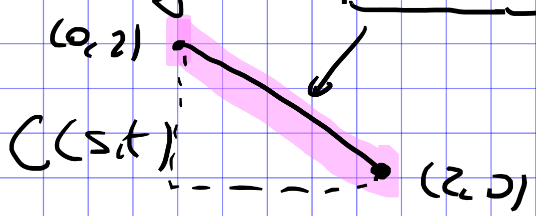
2) Linear equations

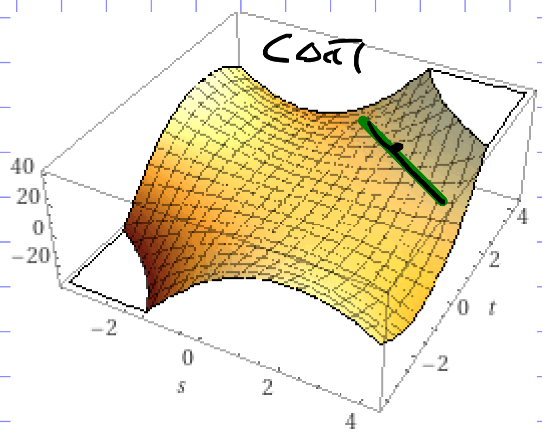
1) We saw in the last lecture the example of optimising a two variable function

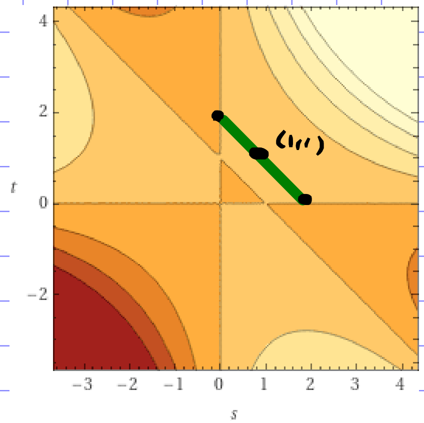
$$C(s,t)$$

under the constraint that (s,t) belongs to a line

$$g(s,t) = \boxed{s+t = 2}$$







Optimising functions when (x,y) are restricted to a constraint of the type

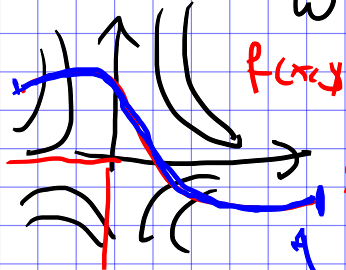
$$\left[\, g(x,y) = C \,\right]$$

Example: In the example above we have optimised

$(s, 2-s)$ $\quad C(s,t)$ constrained to $\boxed{s+t=2}$. $\big(C(s, 2-s) \quad s \in [0,2] \big)$

In this case we could write $\boxed{t = 2-s}$ (t as a function of s)
but can we attack this problem in general, when we have a level curve

$$\left[\, g(x,y) = C \,\right] \; ? \qquad \text{e.g. } \boxed{x^2y^2 + xy^3 = 1}$$



We define the Lagrangian

$$\left[\, \mathcal{L}(x,y) = f(x,y) + \lambda\big(g(x,y) - C\big) \,\right]$$

Note

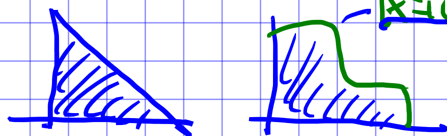
For all $\boxed{(x,y) \text{ such that } g(x,y) = C}$, $\mathcal{L}(x,y) = f(x,y)$

Problem of optimising $f(x,y)$ subjected to $\boxed{g(x,y)=c}$

1) Write the Lagrangian

2) Solve $\left\{\begin{array}{l} \mathcal{L}'_x(x,y)=0 \\ \mathcal{L}'_y(x,y)=0 \\ g(x,y)=c \end{array}\right.$ (1) for $\boxed{(x,y,\lambda)}$

$\boxed{x^2+y^2-2xy=1}$



The triplets solving (1) are the solution candidates

Example Maximise $f(x,y)=x^2-y^2$ subject to the constraint $\boxed{2x+y=5}$,

$$\boxed{\mathcal{L}(x,y)=x^2-y^2+\lambda(2x+y-5)}$$

$$\left.\begin{array}{l} \mathcal{L}'_x(x,y)=2x+2\lambda=0 \\ \mathcal{L}'_y(x,y)=-2y+\lambda=0 \\ \boxed{2x+y=5} \end{array}\right\}$$

$$\left.\begin{array}{ll} (1) & 2x+0y+2\lambda=0 \\ (2) & 0\cdot x-2y+\lambda=0 \\ (3) & 2x+y+0\lambda=5 \end{array}\right\}$$ 3 equations with 3 unknowns

$$\left\{\begin{array}{l} y=5-2x \\ f(x,5-2x) \end{array}\right.$$

From (1) & (2) $\Rightarrow$ $\boxed{x = -\lambda}$ and $\boxed{y = \frac{1}{2}\lambda}$

then (3) yields $2(-\lambda) + (\frac{1}{2}\lambda) = 5 \iff (-2+\frac{1}{2})\lambda = 5$

$\iff -\frac{3}{2}\lambda = 5 \iff \boxed{\lambda = -\frac{10}{3}}$

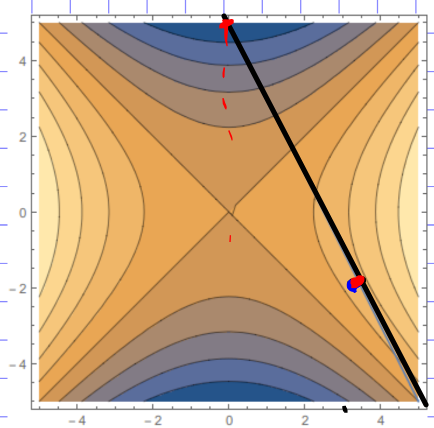
$-3\lambda = 10 \iff -\lambda = \frac{10}{3}$

Hence

$$\begin{cases} x = -(-\frac{10}{3}) = \frac{10}{3} \\ y = \frac{1}{2}(\frac{-10}{3}) = -\frac{5}{3} \end{cases}$$

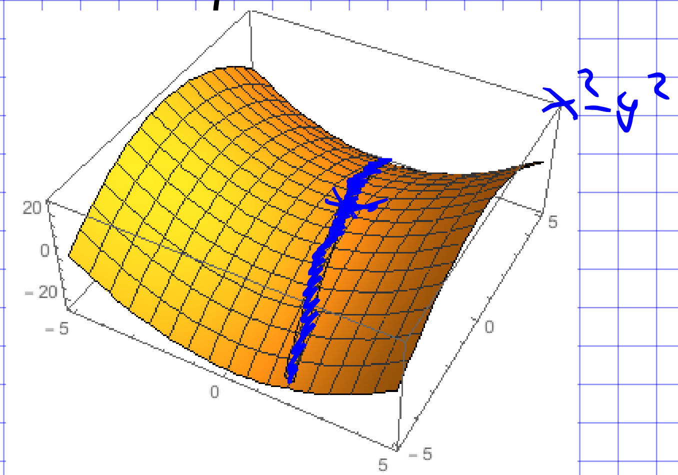
$(\frac{10}{3}, \frac{-5}{3})$ is a solution candidate

$f(\frac{10}{3}, \frac{-5}{3}) = \frac{10^2}{9} - \frac{(-5)^2}{9} = \frac{5^2(4-1)}{9} = \frac{25}{3}.$



$(10/3, -5/3)$

$\boxed{2x+y=5}$



Linear equations

A linear equation with n unknowns is an equation of the form

$$a_1 x_1 + \dots + a_n x_n = b$$

$x_1, \dots, x_n$ unknowns
$a_1, \dots, a_n, b$ given constants

Ex. $2x + y = 1$

We can rewrite this in form of matrices(*) as

$$\underbrace{(a_1 \; a_2 \dots a_n)}_{\substack{\text{A table of 1 row} \\ \text{and n-columns} \\ \boxed{1 \times n}}} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\substack{\text{A table of} \\ \text{n rows and} \\ \text{1 column} \\ \boxed{n \times 1}}} = b$$

A number is a table of 1 row and 1 column.

$$2x + y = 1 \iff (2 \;\; 1) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 1$$

(*) A matrix of m-rows and n-columns is a table on $m \times n$-numbers

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

a_{ij} — i = row, j = column.

Examples of linear equations

1-equation, 2 unknowns $\qquad 2x+3y=1 \qquad // \qquad \left[\begin{pmatrix} 2 & 3 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = 1\right]$

1 equation, 3 unknowns $\qquad 2x+3y+4z=1 \qquad // \qquad \begin{pmatrix} 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1$

Sometimes we are interested in solving several linear equations at the same time. We say that we have a system of m-equations and n-unknowns:

Example

2-equations and 2 unknowns

$$\begin{cases} 2x+y=1 \\ 3x+4y=2 \end{cases} \qquad // \qquad \begin{aligned} \begin{pmatrix} 2 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} &= 1 \\ \begin{pmatrix} 3, & 4 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} &= 2. \end{aligned}$$

b)
$$\left.\begin{aligned} 2x \qquad\quad +2\lambda &= 0 \\ -2y+\lambda &= 0 \\ 2x+y \qquad\quad &= 5 \end{aligned}\right\}$$

3 equations

3 unknowns (x, y, λ)

$$\begin{pmatrix} 2 & 0 & 2 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & -2 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} = 0$$

$$\begin{pmatrix} 2, & 1, & 0 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5$$

In general

$$\begin{cases} a_{11} x_1 + \dots + a_{1n} x_n = b_1 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n = b_m \end{cases}$$

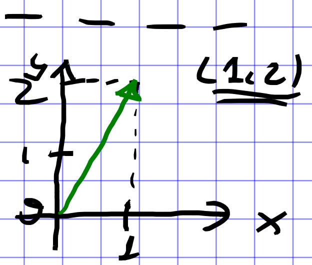
$\boxed{\begin{array}{l} m\text{-equations} \\ n\text{-unknows} \end{array}}$

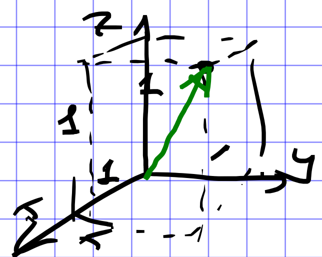
We can collect in a table of the coefficients, unknows

$$\underset{m \times n}{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}} \qquad \underset{n \times 1}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} \qquad \underset{m \times 1}{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}$$

A $m \times 1$ matrix is a (column) vector $\equiv \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (3×1)

A $1 \times n$ matrix is a (row) vector $\equiv (1 \;\; 2 \;\; 3)$ (1×3)



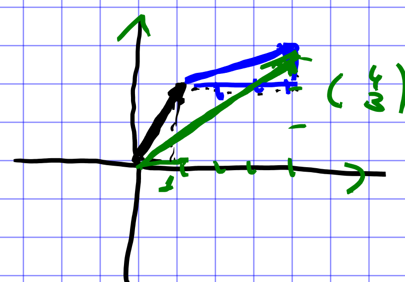


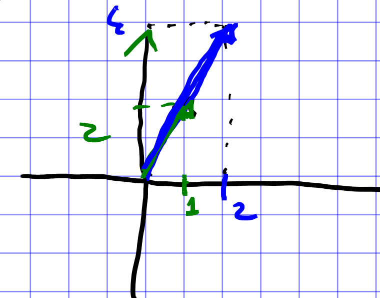
$(1, 1, 1)$

Operation on matrices

(Column) Vectors $(m \times 1)$

- $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+3 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ We can add two vectors





$2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

- $\left[\lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda \end{pmatrix} \right]$

we can multiply a vector by a constant

Same if we have row-vectors $(1 \times n)$

$\left[(1 \;\; 2) + (3 \;\; 1) = (4 \;\; 3) \quad ; \quad 2(1 \;\; 2) = (2 \;\; 4) \right]$

Whenever we have two matrices of the same size, we can add them together (same # of rows and columns)

$$\underbrace{\begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{bmatrix}}_{(m \times n)} \pm \underbrace{\begin{bmatrix} b_{1,1} & \dots & b_{1,n} \\ b_{2,1} & \dots & b_{2,n} \\ \vdots & \vdots & \vdots \\ b_{m,1} & \dots & b_{m,n} \end{bmatrix}}_{(m \times n)} = \underbrace{\begin{bmatrix} a_{1,1} \pm b_{1,1} & \dots & a_{1,n} \pm b_{1,n} \\ a_{2,1} \pm b_{2,1} & \dots & a_{2,n} \pm b_{2,n} \\ \vdots & \vdots & \vdots \\ a_{m,1} \pm b_{m,1} & \dots & a_{m,n} \pm b_{m,n} \end{bmatrix}}_{(m \times n)}$$

Given an scalar c (a real number) and a matrix A $(m \times n)$

$$A = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{bmatrix}$$ we define $$cA = \begin{bmatrix} ca_{1,1} & \dots & ca_{1,n} \\ ca_{2,1} & \dots & ca_{2,n} \\ \vdots & \vdots & \vdots \\ ca_{m,1} & \dots & ca_{m,n} \end{bmatrix}$$

Given a row-matrix $(1 \times \boxed{n})$ $(a_1 \; \dots \; a_n) = A$

and a column matrix $(\boxed{n} \times 1)$ $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = B$

one defines its (escalar) product as

$$\underset{1\times n \cdot\cdot n \times 1}{A \cdot B} = \underset{1 \times n}{(a_1 \; \dots \; a_n)} \cdot \underset{n \times 1}{\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}} = \underset{1 \times 1 \text{ matrix}}{(a_1 \cdot b_1 + \dots + a_n \cdot b_n)}$$

Given a $m \times n$ matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ — Row 1

and $n \times k$ matrix $B = \begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nk} \end{pmatrix}$

column 1.

one defines the $m \times k$ matrix

$$A \cdot B =: C = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & & \vdots \\ c_{m1} & \cdots & c_{nk} \end{pmatrix}$$

where

$$c_{ij} = \underbrace{(a_{i1} \cdots a_{in})}_{i\text{-Row of } A} \cdot \underbrace{\begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}}_{j\text{-column of } B} = a_{i1} \cdot b_{1j} + \cdots + a_{in} \cdot b_{nj}$$

Example

$$A = \begin{pmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

3×2 $\qquad$ 2×2.

$B \cdot A$ is not defined

$$A \cdot B = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}$$

$3 \times 2 =$ matrix

$c_{11} = (3\ 2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3 - 2 = 1$

$c_{12} = (3\ 2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3$

$c_{21} = (-2\ 0) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2$

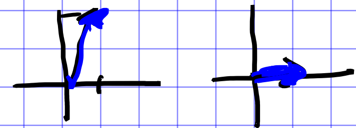
$c_{22} = (-2\ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -2$

$c_{31} = (1\ 3) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2$

$c_{32} = (1\ 3) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$

$$A \cdot B = \begin{pmatrix} 1 & 3 \\ -2 & -2 \\ -2 & 1 \end{pmatrix}$$

Two vectors $\begin{pmatrix} u \\ v \end{pmatrix}$ $\begin{pmatrix} a \\ b \end{pmatrix}$ are equal if each component are equal $\quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \iff u = a \ \underline{\text{and}} \ v = b.$$

Example

- $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \iff b_1 = 1$ and $b_2 = 2$.

- $\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \iff \begin{pmatrix} 2x + 3y \\ x + y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \iff \begin{matrix} 2x + 3y = b_1 \\ \text{and} \\ x + y = b_2 \end{matrix}$

  $\boxed{2} \times 2 \qquad 2 \times \boxed{1} \qquad 2 \times 1$

  $\iff \begin{cases} 2x + 3y = b_1 \\ x + y = b_2 \end{cases}.$

So solving the system of equations

$$\begin{cases} a_{11} x_1 + \ldots + a_{1n} x_n = b_1 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1} x_1 + \ldots + a_{mn} x_n = b_m \end{cases}$$

corresponds to.

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{b}$$

which can be written in matricial form as

$\boxed{A \cdot X = b}$ and the problem becomes finding X

Example The sytems of linear equations

$$\begin{cases} 2x + y = 1 \\ x + y = 2 \end{cases} \qquad \begin{cases} 2x + y + z = 1 \\ 0 \cdot x + y + z = 0 \\ -x + y + 0 \cdot z = 0 \end{cases} \qquad \begin{cases} 2x + y = 1 \\ 3x + y = 1 \\ 4x + 2y = 2 \end{cases}$$

can be written as

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

We define as I_n the $n \times n$ matrix with 1 on the diagonal and zero elsewhere ($I_n \equiv$ identity matrix)

Ex.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Properties of matrices

i) $A \cdot (B + C) = A \cdot B + A \cdot C$ Distributive law.

ii) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ Associative law

iii) In general $AB \neq BA$ even if both products are defined!!!

iv) $I_n \cdot \underset{n \times m}{A} = A$; $A \cdot I_m = A$.

$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ 2×2

$B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ 2×2

$(2\ \ 4)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2 \cdot 1 + 4 \cdot 2 = 10$

$$\underset{2\times 2}{A \cdot B} = \left(\begin{array}{c|c} 7 & 4 \\ \hline 10 & 6 \end{array}\right)$$

$$\neq$$

$$B \cdot A = \begin{pmatrix} 3 & 7 \\ 4 & 10 \end{pmatrix}$$

$A \cdot B \neq B \cdot A$ because $7 \neq 3$

Def. Two matrices $A = (a_{ij})$ $B = (b_{ij})$ are equal if and only if

1) They have the same dimensions.

2) $a_{ij} = b_{ij}$ for all i, j

Transpose

Given a column vector $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ we define its transpose

as $v^T = \begin{pmatrix} 1 & 2 \end{pmatrix}$

or a row vector $u = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, its transpose

is $u^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

In general given a $\boxed{m \times n}$ matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

we define its transpose as the $n \times m$-matrix

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ \vdots & & & \vdots \\ a_{1n} & & \cdots & a_{mn} \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Note

1) $(A^T)^T = A$.   2) $(A \cdot B)^T = B^T \cdot A^T$

3) $(A+B)^T = A^T + B^T$ 4) $(\alpha A)^T = \alpha A^T$

A matrix is called symmetric if $A^T = A$

Example

$f(x,y) = x^2 + xy + y^2$

$$\begin{cases} f'_x = 2x + y \\ f'_y = x + 2y \end{cases}$$

$f''_{xx} = 2$ $\quad f''_{yx} = 1$

$f''_{xy} = 1$ $\quad f''_{yy} = 2$

$$H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$H^T = H$

Solving a system of linear equations

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

corresponds to solve $A \cdot X = b$

$$\underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{b}$$

If we take X for granted, we can write this on a compact form

$$\left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

(Augmented coefficient matrix)

Strategy to solve it

symplify the system by using the following operations

Elementary operations

- Exchange two equations
- Multiply equations by scalars
- Adding a multiple of one equation to another.

Example

$$\begin{cases} 5x - 3y = 25 \\ x + y = 6 \end{cases} \underset{Eq_1 \leftrightarrow Eq_2}{\sim} \begin{cases} x + y = 6 \\ 5x - 3y = 25 \end{cases}$$

$$\underset{-5eq_1 + eq_2 \to eq_2}{\sim} \begin{cases} x + y = 6 \\ -8y = -5 \end{cases}$$

$$\underset{-\frac{1}{8}eq_2 \to eq_2}{\sim} \begin{cases} x + y = 6 \\ \boxed{y = 5/8} \end{cases}$$

$$\underset{eq_1 - eq_2 \to eq_1}{\sim} \begin{cases} x = 6 - 5/8 = \frac{43}{8} \\ y = 5/8 \end{cases}$$

$$\begin{cases} x = 43/8 \\ y = 5/8 \end{cases}$$

Takes time to write the unknowns!

$$\left(\begin{array}{cc|c} 5 & -3 & 25 \\ 1 & 1 & 6 \end{array}\right) \xrightarrow{Row_1 \leftrightarrow Row_2} \left(\begin{array}{cc|c} 1 & 1 & 6 \\ 5 & -3 & 25 \end{array}\right)$$

$$\xrightarrow[\boxed{-5Row_1 + Row_2 \to Row_2}]{} \left(\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & -8 & -5 \end{array}\right)$$

$$\xrightarrow[-\frac{1}{8}Row_2 \to Row_2]{} \left(\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 5/8 \end{array}\right)$$

$$\xrightarrow[Row_1 - Row_2 \to Row_1]{} \left(\begin{array}{cc|c} 1 & 0 & 43/8 \\ 0 & 1 & 5/8 \end{array}\right)$$

$$\boxed{\begin{array}{l} x = 43/8 \\ y = 5/8 \end{array}}$$

Example

$$\begin{cases} 2x + 2z = 0 \\ -2y - z = 0 \\ 2x + y = 5 \end{cases}$$

Example

$$\begin{cases} 2x + y = 1 \\ 4x + 2y = 3 \end{cases}$$

Example

$$\begin{cases} 2x + y = 1 \\ 4x + 2y = 2 \end{cases}$$