Course MM1005

Lecture 13: Matrices and Linear Systems-I

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Questions?

Lecture Goal and Outcome

Goals:

- Introduce the notion of vectors and matrices, their operations, and the connection to linear systems.
- Introduce the Gauss elimination method to solve linear systems.

Learning Outcome: At the end of the lecture you will be able to solve problem like the following:

Problem

The demand and supply equations of a good are given by

$$4P = -Q_d + 240, \qquad 5P = Q_s + 30,$$

where P is the price of the good, Q_d is the quantity demanded, and Q_s is the quantity supplied to the market.
Find the values of P, Q_s, Q_f such that the market is in equilibrium.

Why you should care

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The Portfolio problem

Given a wealth W, we want to invest it in a collection of n risky assets $\{S_1, \ldots, S_n\}$: a fraction of our wealth, w_i say, in asset S_i.
Assuming that short selling is allowed (some of the w_i may be negative (borrow)).

> *How can we choose an optimal set of weights* $\{w_1, ,,,, w_n\}$*, so that our overall investment is likely to yield a promising return with minimal risk?*

To attack this problem, you need to learn several mathematical tools: Linear Algebra, Optimisation and Probability.

Lecture Plan

- Vectors & Matrices
- Matrix operations
- System of linear equations & Gaussian Elimination

Section 1
Vectors and Matrices

Vectors

We can represent a portfolio of n-stocks (or other assets) with a table of n rows and 1 column (vector) as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

where x_i shares of stock $i = 1 \ldots, m$.

Vectors

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$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

where x_i shares of stock $i = 1 \ldots, m$. We can define its transpose as

$$\vec{x}^T = \begin{bmatrix} x_1 & x_2 & \ldots & x_m \end{bmatrix}$$

Similarly, we can write

$$\vec{s}^T = \begin{bmatrix} s_1 & s_2 & \dots & s_n \end{bmatrix}$$

where s_i represents the price of a share of asset i under a particular economic scenario (or the)

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The overall value of the portfolio under the given scenario is

$$s_1 x_1 + s_2 x_2 + \dots + s_n x_n$$

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Multiplication of a $(1 \times n)$ row vector and and $(n \times 1)$ column vector

$$\vec{s}^T \vec{x} := \begin{bmatrix} s_1 & s_2 & \dots & s_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = s_1 x_1 + s_2 x_2 + \ldots + s_n x_n$$

Examples

- $\vec{s}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\vec{x}^T = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}$

Suppose now that we have m different scenarios, each with different price for share i:

$$\vec{s_j}^T = \begin{bmatrix} s_{j1} & s_{j2} & \dots & s_{jn} \end{bmatrix} \qquad j = 1, \dots, m,$$

We can represent all of them in a rectangular array of numbers with m rows and n columns ($m \times n$-Matrix):

$$S = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} = \begin{bmatrix} \vec{s_1}^T \\ \vec{s_2}^T \\ \vdots \\ \vec{s_m}^T \end{bmatrix}$$

The overall value of the portfolio for the m scenarios is given by the $m \times 1$-matrix

$$S\vec{x} := \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \vec{s_1}^T \vec{x} \\ \vec{s_2}^T \vec{x} \\ \vdots \\ \vec{s_m}^T \vec{x} \end{bmatrix}$$

Examples

$$\begin{bmatrix} 3 & -7 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 7 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix},$$

Suppose now, that we want to manage k-different portfolios, which we could represent now as a $n \times k$-Matrix

$$X := \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \dots & \vec{x}_k \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nk} \end{bmatrix}$$

The overall value of the k different portfolios under m different scenarios is given by the $m \times K$ matrix

$$S \cdot X = \begin{bmatrix} Sx_1 & \dots Sx_k \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nk} \end{bmatrix}$$

Multiplication of a $(m \times n)$ Matrix S and and $(n \times k)$ Matrix X

$$\begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} \cdot \begin{bmatrix} {\color{blue}x_{11}} & {\color{red}x_{12}} & \dots & {\color{brown}x_{1k}} \\ {\color{blue}x_{21}} & {\color{red}x_{22}} & \dots & {\color{brown}x_{2k}} \\ {\color{blue}\vdots} & & \ddots & {\color{brown}\vdots} \\ {\color{blue}x_{n1}} & {\color{red}x_{n2}} & \dots & {\color{brown}x_{nk}} \end{bmatrix}$$

gives a $m \times k$-matrix C where the element in the row i and column j is given by the matrix multiplication of the row i of S and the j column of X

$$c_{ij} = \begin{bmatrix} {\color{blue}s_{i1}} & {\color{red}s_{i2}} & \dots & {\color{brown}s_{in}} \end{bmatrix} \cdot \begin{bmatrix} {\color{blue}x_{1j}} \\ {\color{red}x_{2j}} \\ \vdots \\ {\color{brown}x_{nj}} \end{bmatrix} = {\color{blue}s_{i1} x_{1j}} + {\color{red}s_{i1} x_{2j}} + \dots + {\color{brown}s_{in} x_{nk}}$$

Examples

$$\begin{bmatrix} 3 & -7 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 8 & 4 \\ 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix},$$

Comparison of two matrices

Two matrices $(a_{ij})_{m\times n}$, $(b_{ij})_{k\times l}$ are equal if

1. Have the same order (same number of rows and columns) (i.e. $m = k$, $n = l$)
2. For all $i = 1, \dots, m$ and $j = 1, \dots, n$

$$a_{ij} = b_{ij},$$

Examples

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

For which values of x_1, x_2, x_3 are the two following matrices equal?

$$\begin{bmatrix} 4 & x_2 \\ x_3 & 4 \end{bmatrix} = \begin{bmatrix} x_1 & 3 \\ 5 & x_4 \end{bmatrix}$$

Sum of matrices

Whenever we have to matrices of the same order (same number of rows and columns) we can add them together term by term:

Given S, X two $m \times n$ matrices, we define $S + X$ as

$$\begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} & s_{m2} & \dots & s_{mn} \end{bmatrix} + \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mk} \end{bmatrix}$$

$$= \begin{bmatrix} s_{11} + x_{11} & s_{12} + x_{12} & \dots & s_{1n} + x_{1n} \\ s_{21} + x_{21} & s_{22} + x_{22} & \dots & s_{2n} + x_{2n} \\ \vdots & & \ddots & \vdots \\ s_{m1} + x_{m1} & s_{m2} + x_{m2} & \dots & s_{mn} + x_{mn} \end{bmatrix}$$

Product by an scalar

Given an $m \times n$ matrix S, and a real value a we define aS as

$$\begin{bmatrix} as_{11} & as_{12} & \dots & as_{1n} \\ as_{21} & as_{22} & \dots & as_{2n} \\ \\ \vdots & & \ddots & \vdots \\ as_{m1} & as_{m2} & \dots & as_{mn} \end{bmatrix}$$

Examples:

$$3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 7 \end{bmatrix},$$

Transpose

If $\vec{x} = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}$ its transpose is $\vec{x}^T = \begin{bmatrix} x_{11} & x_{21} & \dots & x_{n1} \end{bmatrix}$

In general

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nk} \end{bmatrix} \text{ then } X^T = \begin{bmatrix} x_{11} & x_{21} & \dots & x_{n1} \\ x_{12} & x_{22} & \dots & x_{n2} \\ \vdots & & \ddots & \vdots \\ x_{1k} & x_{2k} & \dots & x_{nk} \end{bmatrix}$$

If A, B are $m \times n$-matrices, C $n \times k$-matrix, and $a \in \mathbb{R}$:

1. $(A^T)^T = A$
2. $(AC)^T = C^T A^T$
3. $(A + B)^T = A^T + B^T$
4. $(aA)^T = aA^T$

Section 2
Linear Systems

Linear systems: Motivation-I

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The matrix S below represents the net profits to 3 stocks under 4 outcomes or scenarios:

$$S := \begin{bmatrix} -2 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 6 \\ 1 & -1 & -5 \end{bmatrix}$$

Can we invest in a way that produces the vector $\vec{v} = \begin{bmatrix} 0 & 0 & 0 & 5 \end{bmatrix}^T$?
How can we figure out the answer to this?
We look for a vector $\vec{x} = \begin{bmatrix} x & y & z \end{bmatrix}^T$ such that

$$\begin{bmatrix} -2 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 6 \\ 1 & -1 & -5 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{Unknown}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \Leftrightarrow \left\{ \begin{array}{rcl} -2x + 3y + z & = & 0 \\ -x + 2y & = & 0 \\ -y + 6z & = & 0 \\ x - y - 5z & = & 5 \end{array} \right.$$

Linear systems: Motivation-II

The demand and supply equations of a good are given by

$$4P = -Q_d + 240, \qquad 5P = Q_s + 30,$$

where P is the price of the good, Q_d is the quantity demanded, and Q_s is the quantity supplied to the market.
At the point of intersection of the demand and supply curves (i.e. $Q_s = Q_p =: Q$), the market is said to be in equilibrium because the quantity demanded is equal to the quantity supplied.
How can we find that equilibrium point?
We need to find (P, Q) such that

$$\begin{cases} 4P + Q &= 240 \\ 5P - Q &= 30 \end{cases} \Leftrightarrow \begin{bmatrix} 4 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 240 \\ 30 \end{bmatrix}$$

Linear systems

A linear system of m-(linear) equations and n-unknowns $x_1, \ldots x_n$

$$\left\{\begin{array}{c} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_1 \end{array}\right. \quad \Leftrightarrow A\vec{x} = \vec{b},$$

where

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

A is called the coefficient matrix $\quad \vec{x}$ unknown $\quad \vec{b}$ independent term.

Taking the unknowns for granted, we can write the system in a compact form (Augmented coefficient matrix):

$$A\vec{x} = \vec{b} \Leftrightarrow \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right]$$

Examples

$$\begin{bmatrix} 4 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 240 \\ 30 \end{bmatrix} \Leftrightarrow \left[\begin{array}{cc|c} 4 & 1 & 240 \\ 5 & -1 & 30 \end{array}\right]$$

$$\begin{bmatrix} -2 & 3 & 1 \\ -1 & 2 & 0 \\ 0 & -1 & 6 \\ 1 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 6 & 0 \\ 1 & -1 & -5 & 5 \end{array}\right]$$

Simple systems $Ax = b$

The equation...

1. $1 \cdot x = 1$ has a unique solution $x = 1$.

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2. The equation $0 \cdot x = 0$ is satisfied for all $x \in \mathbb{R}$. Has an infinite number of solutions.

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The system of equations...

1. $\left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 30 \end{array}\right] \Leftrightarrow \begin{cases} x = 10 \\ y = 30 \end{cases}$ has only one solution.

Simple systems $Ax = b$

The equation...

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The system of equations...

1. $\left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 30 \end{array}\right] \Leftrightarrow \begin{cases} x = 10 \\ y = 30 \end{cases}$ has only one solution.
2. $\left[\begin{array}{cc|c} 1 & 1 & 10 \\ 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x + y = 10 \\ 0 \cdot x + 0 \cdot y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 10 - t \\ y = t \end{cases} \quad t \in \mathbb{R},$
   The system has an infinite number of solutions.

Simple systems $Ax = b$

The equation...

1. $1 \cdot x = 1$ has a unique solution $x = 1$.
2. The equation $0 \cdot x = 0$ is satisfied for all $x \in \mathbb{R}$. Has an infinite number of solutions.
3. $0 \cdot x = 1$ has no solution.

The system of equations...

1. $\left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 30 \end{array}\right] \Leftrightarrow \begin{cases} x = 10 \\ y = 30 \end{cases}$ has only one solution.
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   The system has an infinite number of solutions.
3. $\left[\begin{array}{cc|c} 1 & 1 & 10 \\ 0 & 0 & 1 \end{array}\right] \Leftrightarrow \begin{cases} x + y = 10 \\ 0 = 1 \end{cases}$ has no solution.

Gauss elimination method

Strategy: Simplify the system by using the following elementary operations:

1. Exchange two equations (rows);

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Example

$$\left[\begin{array}{cc|c} 5 & -3 & 25 \\ 1 & 1 & 6 \end{array}\right]$$

Gauss elimination method

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Example

$$\left[\begin{array}{cc|c} 5 & -3 & 25 \\ 1 & 1 & 6 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 5 & -3 & 25 \end{array}\right]$$

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$$\underset{R_2+8R_1 \to R_1}{\sim} \left[\begin{array}{cc|c} 8 & 0 & 43 \\ 0 & -8 & -5 \end{array}\right]$$

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Strategy: Simplify the system by using the following elementary operations:

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3. Adding a multiple of one equation to another

Example

$$\left[\begin{array}{cc|c} 5 & -3 & 25 \\ 1 & 1 & 6 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 5 & -3 & 25 \end{array}\right] \underset{-5R_1+R_2 \to R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & -8 & -5 \end{array}\right]$$

$$\underset{R_2+8R_1 \to R_1}{\sim} \left[\begin{array}{cc|c} 8 & 0 & 43 \\ 0 & -8 & -5 \end{array}\right] \underset{\substack{R_1/8 \to R_1 \\ -R_2/8 \to R_2}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 43/8 \\ 0 & 1 & 5/8 \end{array}\right]$$

If we want to solve a bunch of systems with the same coefficients, but different independent terms

$$Ax = \vec{b}_1, \quad Ax = \vec{b}_2 \ldots \quad Ax = \vec{b}_k$$

we can run the method for all of them at the same time by writing

$$\left[\begin{array}{c|cccc} A & b_1 & b_2 & \ldots & b_k \end{array}\right]$$

Example Determine if any of following systems has any solution. If so, find all solutions.

1. $\begin{cases} 2x + y = 1 \\ 4x + 2y = 3 \end{cases}$
2. $\begin{cases} 2x + y = 1 \\ 4x + 2y = 2 \end{cases}$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 1 \\ 4 & 2 & 3 & 2 \end{array}\right] \underset{-2R_1+R_2 \to R_2}{\sim} \left[\begin{array}{cc|cc} 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Exercises

Determine if the systems of the motivational examples have any solution. If so, find the solution(s):

1. $\left[\begin{array}{cc|c} 4 & 1 & 240 \\ 5 & -1 & 30 \end{array}\right]$. $P = 30$, $Q = 120$
2. $\left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 6 & 0 \\ 1 & -1 & -5 & 5 \end{array}\right]$

Multicomodity market

The demand and supply functions for two interdependent commodities are given by

$$
\begin{aligned}
Q_{d1} &= a_1 + b_1 P_1 + c_1 P_2 \\
Q_{d2} &= a_2 + b_2 P_1 + c_2 P_2 \\
Q_{s1} &= a_3 + b_3 P_1 \\
Q_{s2} &= a_3 + c_3 P_2
\end{aligned}
$$

where P_i, Q_{di} and Q_{ds} denote the price and quantity demanded, and quantity supplied for the ith good, and a_i, b_i, c_i are constants depending on the model.
Determine the equilibrium price and quantity for this two-commodity model (i.e. $Q_{di} = Q_{si}$)

Thank you for your attention!