Course MM1005

Lecture 14: Matrices and Linear Systems-II

Sofia Tirabassi
tirabassi@math.su.se

Salvador Rodriguez Lopez
s.rodriguez-lopez@math.su.se

Questions?

Lecture Goal and Outcome

Stockholm University

Goals:

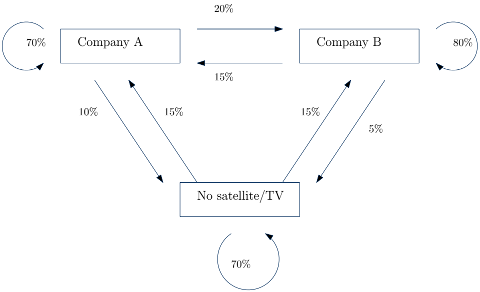
- Calculate determinants of Order 2 and 3 by cofactors.
- Determine whether a matrix is invertible or not and calculate its inverse.
- Determine the number of solutions of a linear system of equations.

Learning Outcome: At the end of the lecture you will be able to solve problem like the following:
Determine for which values of the parameter a, the system

$$\begin{cases} ax + y + 3z = 2 \\ 2x + y + az = 2 \\ 2x + y + 3z = a \end{cases}$$

has exactly one solution, no solution, or infinitely many solutions.

Why you should care



A Consumer Preference Model

Two competing companies offer satellite television service to a city with 100000 households. The figure shows the changes in satellite subscriptions/year.
Company A now has $15,000$ subscribers and Company B has $20,000$ subscribers. How many subscribers will each company have in one year?

Inverse of a square matrix

A matrix with the same amount of rows and columns is called a **square matrix**.
We say that a $n \times n$ matrix A is **invertible** if there exists a matrix B such that

$$AB = BA = \mathbf{I_n}$$

where $\mathbf{I_n}$ is the identity matrix

$$\mathbf{I_n} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

If that matrix exists, it is unique, is called the **inverse** of A, and it is denoted by A^{-1}.

Determinant of a matrix

The linear equation $ax = b$ has a unique solution for all b if, and only if, a^{-1} exists (a is invertible), which is equivalent to $a \neq 0$.
Recall that a^{-1} is the only real number such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1,$$

For a given $a \in \mathbb{R}$, the 1×1 matrix $[a]$ is invertible if $\det[a] = a \neq 0$ and in that case

$$[a]^{-1} = [a^{-1}],$$

What can we say for $n = 2$?

Given a 2×2-matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ we define its determinant as $\det A = a_{11} a_{22} - a_{21} a_{12}$.

Example: If $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, $\quad B = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ then $B = A^{-1}$ because

$$A \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B \cdot A$$

Moreover $\det A = 1 \cdot 3 - 2 \cdot 1 = 3$, $\quad \det B = ?$

A 2×2-matrix A is invertible, if and only if, $\det A \neq 0$

Note that if A is invertible, for all vector $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, the linear system of equations $Ax = b$ has a unique solution

$$Ax = b \Leftrightarrow A^{-1}Ax = A^{-1}b \Leftrightarrow x = A^{-1}b,$$

Example

1. $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

If $\det(A) \neq 0$, the linear system $Ax = b$ has a unique solution: $x = A^{-1}b$.
If $\det(A) \neq 0$, the linear system $Ax = b$ has either no solution, or an infinite number of solutions.
(we can use Gaussian elimination to find the solution(s))

How can we find the inverse?

If $\det A \neq 0$, we want to find B such that

$$A \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Leftrightarrow A \cdot \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \& A \cdot \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So, we need to solve two systems of Linear equations with the same independent coefficients (A)! So we can run the Gauss-elimination at the same time by writting:

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right]$$

Example By making $-2R_1 + R_2 \to R_2$ and $-R_2 + R_1 \to R_1$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

So we have that

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

We could have used the same principle to find a 2×4-matrix M such that

$$A \cdot M = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 5 \end{bmatrix}$$

wit the same row operations

$$\left[\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|cccc} 1 & 0 & 3 & -1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 1 & 1 \end{array}\right]$$

Square matrices $n = 3$ and higher

Stockholm University

Given a 3×3-matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ we define

$$\det A := a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A similar definition can be given in any dimension $n \geq 3$. We will restrict ourselves to $n = 1, 2, 3$ in this course.

Stockholm University

A $n \times n$-matrix A is invertible, if and only if, $\det A \neq 0$.
If $\det(A) \neq 0$, the linear system $Ax = b$ has a unique solution: $x = A^{-1}b$.
If $\det(A) \neq 0$, the linear system $Ax = b$ has either no solution, or an infinite number of solutions.

The same method as for $n = 2$ works in any dimension. E.g.

Determine whether the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ is invertible, and if so, calculate its inverse:

$$\det A =$$

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ \text{and } -R_1 + R_3 \rightarrow R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \qquad 2R_2 + R_3 \rightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \qquad -R_3 \rightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \qquad \begin{array}{l} 3R_3 + R_2 \rightarrow R_2 \\ -3R_3 + R_1 \rightarrow R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \qquad -2R_2 + R_1 \rightarrow R_1 \quad ,$$

Aug. 2013

For which values of c the system of equations below has a unique solution, infinitely many, or no solutions at all?

$$\begin{cases} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (c^2 - 14)z &= c + 2 \end{cases}$$

2019

Let A be the matrix $\begin{bmatrix} 2 & 0 & 2+k \\ 3 & 1 & 0 \\ k & 0 & -2 \end{bmatrix}$

1. Find $\det A$ as a function of k;
2. Show that A is always invertible for any value of k;
3. Find the minimal and maximal value $\det A$ can take, for $k \in [-2, 2]$.

Thank you for your attention!