Lecture 13

→ Linear systems

- $\boxed{A \cdot X = b}$ ⇝ Gaussian elimination method.
- ($\underline{n \times n}$ matrices $A \to \boxed{\det(A)}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$n=2$ $\quad \det(A) = a_{11} a_{22} - a_{12} \cdot a_{21}$

$n=3$ $\quad \det(A) = a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

- A is invertible ⟺ there exists A^{-1} such that

$$A^{-1} \cdot A = A \cdot A^{-1} = I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \vdots & 1 & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$(I_n \cdot X = X)$

⟺ $\boxed{\det(A) \neq 0}$

- The system $\boxed{A \cdot X = b}$ has $\underline{\text{a unique solution}}$ if $\det(A) \neq 0$

Note $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

in which case

$$\boxed{A \cdot X = b} \iff \overbrace{A^{-1} \cdot A}^{I_n} X = A^{-1} b \iff X = A^{-1} b.$$

If $\det(A) = 0$, it can either have no solution or an infinite number of them

4. For which values of the constant c the system of equations below has a unique solution, infinitely many solutions and no solutions at all?

$$\begin{cases} x+2y-3z=4 \\ 3x-y+5z=2 \\ 4x+y+(c^2-14)z=c+2 \end{cases}$$

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$$A\cdot\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix} 4\\ 2\\ c+2\end{pmatrix} \quad (A\cdot X=b)$$

$$A=\begin{pmatrix} 1 & \boxed{2} & -3\\ 3 & -1 & 5\\ 4 & 1 & c^2-14\end{pmatrix}$$

If $\det(A)\neq 0$, the system has a unique solution

$$\det(A)=1\cdot\det\begin{pmatrix} -1 & 5\\ 1 & c^2-14\end{pmatrix}-2\cdot\det\begin{pmatrix} 3 & 5\\ 4 & c^2-14\end{pmatrix}+(-3)\det\begin{pmatrix} 3 & -1\\ 4 & 1\end{pmatrix}$$

$$=1\cdot(-c^2+14-5)-2\cdot(3c^2-42-20)-3(3-(-1)4)$$

$$=(9-c^2)-2(3c^2-62)-21$$

$$=9-c^2-6c^2+124-21=-7c^2+112$$

Now $\det(A) = 0 \iff -7c^2 + 112 = 0 \iff c^2 = \frac{112}{7} = 16$

$$\iff c = \pm\sqrt{16} = \pm 4.$$

So if $c \neq \pm 4$, then the system has a unique solution.

Last step: Study the system for $\boxed{c = 4}$ and $\boxed{c = -4}$

(We will continue this in this lecture!)

4. For which values of the constant c the system of equations below has a unique solution, infinitely many solutions and no solutions at all?

$$\begin{cases} x + 2y - 3z = 4 \\ 3x - y + 5z = 2 \\ 4x + y + (c^2 - 14)z = c + 2 \end{cases}$$

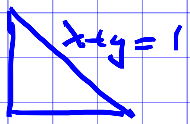
$c = 4$

$c = -4$

$$\left(\begin{array}{ccc|cc} 1 & 2 & -3 & 4 & 4 \\ 3 & -1 & 5 & 2 & 2 \\ 4 & 1 & 2 & \underset{c=4}{6} & -2 \end{array}\right)$$

Lets look first to a simpler example

Solve the equation (find all (x,y) satisfying)



$\boxed{x+y=1}$ (1-equation

(2-unknowns. (we have "1-too-many")

We can write $[x+y=1]$ as $[x=1-y]$

If we let $\begin{cases} y=t \\ x=1-t \end{cases}$ with $t\in\mathbb{R}$, this describes (parametrises)

$(x+y=t+(1-t)=1)$

all possible solutions

_____ . _____ . _____ . _____ .

2-equations
3-unknowns.

$$\boxed{(1)\ \begin{cases} x+z=1 \\ y+z=0 \end{cases}}$$

we could write it as

$$\begin{cases} x=1-z \\ y=-z \end{cases}$$

so if we let $\begin{cases} z=t \\ x=1-t \\ y=-t \end{cases}$ $t\in\mathbb{R}$, this describes all

the solutions of (1)

$$\begin{cases} x+z=t+(1-t)=1 \\ y+z=-t+t=0 \end{cases}$$

$[x+y=1]$

$1 \downarrow \left(1 \; \underset{\text{2 colums}}{1} \mid 1 \right)$ $\quad (1\ 1)\begin{pmatrix} x \\ y \end{pmatrix} = 1$

$[y = t \in \mathbb{R}]$

$(1 \mid 1-t)$

$$\begin{cases} x = 1-t \\ y = t \end{cases}$$

$$\begin{cases} x + z = 1 \\ y + z = 0 \end{cases}$$

$$\left(\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix} \right)$$

$[z = t]$ $t \in \mathbb{R}$

$$\left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 1-t \\ -t \end{matrix} \right)$$

$$\begin{cases} x = 1-t \\ y = -t \end{cases}$$

How many solutions have the following systems

i) $0 \cdot x = 0$ $\quad (0 \mid 0)$

Infinite

ii) $0 \cdot x = 1$ $\quad (0 \mid 1)$

No solution.

iii) $0 \cdot x + 0 \cdot y = 0$

Infinite

$(0 \ 0 \mid 0)$

iv) $0 \cdot x + 0 \cdot y = 1$

No solutions

$(0 \ 0 \mid 1)$

v) $0 \cdot x + 0 \cdot y + 0 \cdot z = 0$

Infinite

$(0 \ 0 \ 0 \mid 0)$

vi) $0 \cdot x + 0 \cdot y + 0 \cdot z = 1$

No solution

$(0 \ 0 \ 0 \mid 1)$

4. For which values of the constant c the system of equations below has a unique solution, infinitely many solutions and no solutions at all?

$$\begin{cases} x + 2y - 3z = 4 \\ 3x - y + 5z = 2 \\ 4x + y + (c^2 - 14)z = c + 2 \end{cases}$$

$c = 4$

$c = -4$

$$\left(\begin{array}{ccc|cc} 1 & 2 & -3 & 4 & 4 \\ 3 & -1 & 5 & 2 & 2 \\ 4 & 1 & 2 & 6 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|cc} 1 & 2 & -3 & 4 & 4 \\ 0 & -7 & 14 & -10 & -10 \\ 0 & -7 & 14 & -10 & -18 \end{array}\right)$$

$$\left[\begin{array}{l} -3R_1 + R_2 \to R_2 \\ -4R_1 + R_3 \to R_3 \end{array}\right]$$

$$\sim \left(\begin{array}{ccc|cc} 1 & 2 & -3 & 4 & 4 \\ 0 & -7 & 14 & -10 & -10 \\ 0 & 0 & 0 & 0 & -8 \end{array}\right)$$

$-R_2 + R_3 \to R_3$

$\boxed{c = -4}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & 0 & -8 \end{array}\right)$$

$\boxed{0 \cdot x + 0 \cdot y + 0 \cdot z = -8}$

The system has no solutions if $c = -4$.

$c=4$ $\quad \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \end{array}\right)$

$[0\cdot x + 0\cdot y + 0\cdot z = 0]$

$\sim \left[-\frac{1}{7}R_3 \to R_3\right] \quad \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \end{array}\right)$

$\sim \left[-2R_3 + R_1 \to R_1\right] \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & -\frac{20}{7}+4 \\ 0 & 1 & -2 & \frac{10}{7} \end{array}\right)$ $\quad 4 - \frac{20}{7} = \frac{28-20}{7} = 8/7$

$\sim \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} & & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 8/7 \\ 0 & 1 & -2 & 10/7 \end{array}\right) \end{array}$

$\begin{cases} x + z = 8/7 \\ y - 2z = 10/7 \end{cases}$ $\quad \begin{array}{|l} x = 8/7 - z \\ y = 10/7 + 2z \end{array}$ $(z \in \mathbb{R})$

Let $z = t$ $\quad t \in \mathbb{R}$.

$\left(\begin{array}{cc|c} 1 & 0 & 8/7 - t \\ 0 & 1 & 10/7 + 2t \end{array}\right)$

$\begin{cases} z = t \\ x = 8/7 - t \\ y = 10/7 + 2t \end{cases}$

There is an infinite number of solutions if $c = 4$.

1. Find all solutions to the following systems of equations:

(a)
$$\begin{aligned} 3y - z + w &= 1 \\ -x + 2y + 2z &= 0 \\ -x + y - 3z + w &= -1 \end{aligned}$$

(b)
$$\begin{aligned} 2A - B - 2C &= -2 \\ 2A + 2B + 4C &= 1 \end{aligned}$$

(2019)

(b) Writing the system using matrices we obtain

$$\begin{pmatrix} 2 & -1 & -2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Which is equivalent to

$$\left(\begin{array}{ccc|c} 2 & -1 & -2 & -2 \\ 2 & 2 & 4 & 1 \end{array}\right)$$

Gaussian elimination gives us the system is equivalent to.

$$(-R_1 + R_2 \to R_2) \quad \left(\begin{array}{ccc|c} 2 & -1 & -2 & -2 \\ 0 & 3 & 6 & 3 \end{array}\right)$$

$$\sim \left[\tfrac{1}{3} R_2 \to R_2\right] \quad \left(\begin{array}{ccc|c} 2 & -1 & -2 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$\underset{R_2 + R_1 \to R_1}{\sim} \left(\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{array}\right) \sim \left(\tfrac{1}{2} R_1 \to R_1\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 2 & 1 \end{array}\right)$$

So we have

$$\begin{cases} x = -\frac{1}{2} \\ y + 2z = 1 \end{cases} \iff \begin{cases} x = -\frac{1}{2} \\ y = 1 - 2z \\ z \in \mathbb{R}. \end{cases} \quad \left(\begin{cases} x = -\frac{1}{2} \\ y = 1 - 2t \\ z = t \end{cases} \quad t \in \mathbb{R} \right)$$

1. Find all solutions to the following systems of equations:

(a)
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(b)
$$\begin{aligned} 2A - B - 2C &= -2 \\ 2A + 2B + 4C &= 1 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 0 & 3 & -1 & 1 & 1 \\ -1 & 2 & 2 & 0 & 0 \\ -1 & 1 & -3 & 1 & -1 \end{array}\right) \sim \left[\begin{array}{c} -R_2 \to R_2 \\ R_2 \leftrightarrow R_1 \end{array}\right] \left(\begin{array}{cccc|c} 1 & -2 & -2 & 0 & 0 \\ 0 & 3 & -1 & 1 & 1 \\ -1 & 1 & -3 & 1 & -1 \end{array}\right)$$

$$\sim \left[R_1 + R_3 \to R_3 \right] \left(\begin{array}{cccc|c} 1 & -2 & -2 & 0 & 0 \\ 0 & 3 & -1 & 1 & 1 \\ 0 & -1 & -5 & 1 & -1 \end{array}\right)$$

$$\sim \left[\begin{array}{c} -R_3 \to R_3 \\ R_2 \leftrightarrow R_3 \end{array}\right] \left(\begin{array}{cccc|c} 1 & -2 & -2 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 \\ 0 & 3 & -1 & 1 & 1 \end{array}\right)$$

$$\sim \left[-3R_2 + R_3 \to R_3 \right] \left(\begin{array}{cccc|c} 1 & -2 & -2 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 \\ 0 & 0 & -16 & 4 & -2 \end{array} \right) \quad \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$\sim \left[\tfrac{1}{4} R_3 \to R_3 \right] \left(\begin{array}{cccc|c} 1 & -2 & -2 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 \\ 0 & 0 & -4 & 1 & -\tfrac{1}{2} \end{array} \right)$$

$$\sim \left[R_3 + R_2 \to R_2 \right] \left(\begin{array}{cccc|c} 1 & -2 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & \tfrac{1}{2} \\ 0 & 0 & -4 & 1 & -\tfrac{1}{2} \end{array} \right)$$

$$\sim \left[2R_2 + R_1 \to R_1 \right] \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & \tfrac{1}{2} \\ 0 & 0 & -4 & 1 & -\tfrac{1}{2} \end{array} \right) \quad \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$\begin{cases} x = 1 \\ y + z = \tfrac{1}{2} \\ -4z + w = -\tfrac{1}{2} \end{cases} \implies \begin{cases} z = t \quad t \in \mathbb{R} \\ x = 1 \\ y = \tfrac{1}{2} - t \\ w = -\tfrac{1}{2} + 4t \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \tfrac{1}{2} - z \\ 0 & 0 & 1 & -\tfrac{1}{2} + 4z \end{array} \right)$$

3. Let A be the matrix $\begin{pmatrix} 2 & 0 & 2+k \\ \cancel{3} & -1 & \cancel{0} \\ k & 0 & -2 \end{pmatrix}$, depending on a parameter $k \in \mathbb{R}$.

(2019)

(a) Find $\det A$, as a function of k.

(b) Show that $\boxed{A \text{ is always invertible}}$ for any value of k.

(c) Find the maximal and minimal values $\det A$ can take, for k in the range $-2 \leq k \leq 2$.

$$\det(A) = 2\underbrace{\det\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}}_{2} - 0 \cdot \underbrace{\det\begin{pmatrix} 3 & 0 \\ k & -2 \end{pmatrix}}_{-6} + (2+k)\underbrace{\det\begin{pmatrix} 3 & -1 \\ k & 0 \end{pmatrix}}_{k}$$

$$= 4 - 0 + (2+k)k = \boxed{4 + 2k + k^2 =: f(k)}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det(A) = (-1)\det\begin{pmatrix} 2 & 2+k \\ k & -2 \end{pmatrix} - \cancel{0 \det\begin{pmatrix} 3 & 0 \\ k & -2 \end{pmatrix}} - \cancel{0 \cdot \det\begin{pmatrix} 2 & 2+k \\ 3 & 0 \end{pmatrix}}$$

b) $k^2 + 2k + 4 = 0 \iff k = \dfrac{-2 \pm \sqrt{4-16}}{2} = \dfrac{-2 \pm \sqrt{-12}}{2}$

• The equation has no real solutions as $\sqrt{-12}$ is not real.

So for all $k \in \mathbb{R}$ $f(k) \neq 0 \iff \det(A) \neq 0 \iff A$ is invertible.

c) • $f(k) = k^2 + 2k + 4$ $\quad k \in [-2, 2]$.

• $f'(k) = 2k + 2 = 0 \iff \boxed{k = -1}$ & $[f(-1) = (-1)^2 - 2 + 4 = 3]$

• $f(-2) = 4 - 4 + 4 = \underline{4}$ $\quad f(2) = 4 + 4 + 4 = 12$.

the maximum value of det(A) on [-2, 2] is 12
and the minimum value on [-2, 2] is 3.

$$\left[\ \deg(P) \leq 1 \qquad \frac{P(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}. \right]$$

4. (a) For which values of c is the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & c \\ 1 & c & 1 \end{pmatrix}$ invertible?

(2018)

(b) Find all solutions to the equation

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

or determine that it has no solutions.

3. Find all solutions of the following systems of equations.

(a)
$$\begin{aligned} 2x - 2y \phantom{{}-3z} + w &= 4 \\ x \phantom{{}-2y} - 3z + w &= 3 \end{aligned}$$

(b)
$$\begin{aligned} x_2 + 3x_3 &= 1 \\ x_1 \phantom{{}+2x_2} + 3x_3 &= 0 \\ x_1 + 2x_2 + 9x_3 &= 1 \end{aligned}$$

(2017)

5. Find all the possible solutions of the following system of linear equations

$$\begin{aligned} x_1 + \ x_2 + \ x_3 + \ x_4 &= 5; \\ x_1 + 2x_2 + 2x_3 + 2x_4 &= 10; \\ x_1 + 2x_2 + 3x_3 + 3x_4 &= 15; \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 20. \end{aligned}$$

First we will write the system of equations in matrix form and we get

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & 2 & 2 & 2 & 10 \\ 1 & 2 & 3 & 3 & 15 \\ 1 & 2 & 3 & 4 & 20 \end{array}\right)$$

We will use Gaussian elimination to find the solutions. First we subtract the first row from the second, third and fourth row and we get

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 2 & 10 \\ 0 & 1 & 2 & 3 & 15 \end{array}\right)$$

Then we subtract the second row from the third and the fourth row and get

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 & 10 \end{array}\right)$$

After that we subtract the third row from the fourth row and get

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right)$$