GREEN'S FORMULA, LINEAR CONTINUITY, AND HAUSDORFF MEASURE(1)

BY

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1. **Introduction.** Let $A(x,y)$ and $B(x,y)$ be defined on a square Q. If one makes strong enough assumptions about $A(x,y)$ and $B(x,y)$ and their partial derivatives, the Green's formula

$$\int_{\partial Q} A\,dx + B\,dy = \iint_Q (B_x - A_y)\,dxdy \tag{1}$$

holds. Cohen [2] and Shapiro [6] have shown that (1) holds with much weaker than the usual assumptions on the functions concerned. In each of these papers $(B_x - A_y)$ is assumed to be in $L_1(Q)$ and the partial derivatives (with respect to both variables) of the functions are assumed to exist except on certain sets. Cafiero [1], in an investigation of a different aspect of the problem, has shown that a similar result holds. The sets where the partial derivatives fail to exist are not the same in each of these papers. In this paper, we will give a proof of (1) which includes all of these results, at the same time we will weaken the continuity requirements on the functions $A(x,y)$ and $B(x,y)$.

In the fourth section, we will give an example which shows that the assumptions we make cannot be weakened substantially.

2. **Preliminaries.** By a square or a rectangle, we will always mean a closed square or rectangle with its sides parallel to the coordinate axes. We will denote the length of the longest side of a rectangle R with $s(R)$.

We will write $|E|$ for the Lebesgue measure of E, using the same notation for both one-dimensional and two-dimensional measure, and we will write $\delta(E)$ for the diameter of E.

We will use $H(p,E)$ to denote the Hausdorff p-dimensional measure of E. That is $H(p,E) = \sup_\varepsilon \inf \{ \sum \delta(O_i)^p;\ \delta(O_i) < \varepsilon\}$ where each O_i is open and $\bigcup O_i \supset E$. $H(E) = H(1,E)$.

Definition. A function $f(x,y)$ is linearly continuous at a point (a,b) if $f(a,y)$ is continuous at b and $f(x,b)$ is continuous at a. $f(x,y)$ is linearly continuous

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on R if it is linearly continuous at each point of R. (We consider functions which are linearly continuous except perhaps at the points of a set E with $H(E)=0$. Goffman considers a more general class of linearly continuous functions in [3], where he shows that such functions are measurable.)

When discussing a particular pair of functions, we will use the phrase "(1) holds in a neighborhood of (a,b)" to mean that there is an open neighborhood G of (a,b) such that (1) holds for every rectangle contained in G.

REMARK 1. If $A(x,y)$ and $B(x,y)$ are bounded and linearly continuous on a rectangle R and if $(B_x - A_y)$ is in $L_1(R)$, then in order to show that for some constant K

$$\left| \int_{\partial R} A\,dx + B\,dy - \iint_R (B_x - A_y)\,dxdy \right| \leqq K,$$

it is sufficient to show that the same inequality holds for every rectangle contained in $\operatorname{int}(R)$.

REMARK 2. Under the conditions of Remark 1, in order to show that (1) holds for R, it is sufficient to show that (1) holds in a neighborhood of each point of $\operatorname{int}(R)$.

Remark 1 follows immediately from the dominated convergence theorem. To see that Remark 2 is true let R' be any rectangle contained in $\operatorname{int}(R)$. Since R' is compact and (1) holds in a neighborhood of each point of R', if we subdivide R' into small enough rectangles, each is contained in a neighborhood in which (1) holds. It follows that (1) holds for R' and hence, by Remark 1, (1) holds for R.

LEMMA 1. *If E is a closed subset of a square Q, $H(E)$ is finite, and m is any positive integer, E can be covered with a finite number of nonoverlapping rectangles R_j such that $s(R_j) < 1/2^m$, $\sum s(R_j) < 8(H(E)+1)$, and $\left|\bigcup R_j\right| < 8(H(E)+1)/2^m$, and such that there is a point of E on each side of each R_j.*

Proof. Assume for simplicity that $s(Q) < 1$, and divide Q into four equal nonoverlapping squares Q_1^k, $k = 1,\cdots,4$. Proceeding in this way construct a sequence of subdivisions of Q into nonoverlapping squares Q_n^k, $k=1,\cdots,4^n$, with $s(Q_n^k) = 1/2^n$.

Since $H(E)$ is finite and E is closed, there is a finite open covering of E with sets O_i, such that $\delta(O_i) < 1/2^{m+1}$ and $\sum\delta(O_i) < (H(E)+1)$. For each O_i, we can find a $k(i)$ such that $1/2^{k(i)+1} \leqq \delta(O_i) < 1/2^{k(i)}$. It is clear that O_i is contained in a square S_i which is the union of four squares of the $k(i)$th subdivision of Q, and that $s(S_i) = 2/2^{k(i)} \leqq 4\delta(O_i) < 2/2^m$.

We may assume that $s(S_i) \geqq s(S_{i+1})$ and set $T_1 = S_1$ and $T_i = S_i - \bigcup_1^{i-1} S_j$ to obtain a new covering of E with nonoverlapping sets T_i, each of which is the union of at most four nonoverlapping sets of the $k(i)$th subdivision of Q.

If we let J_j denote the squares which make up the sets T_i and contain points of E, and if we let R_j be the smallest rectangle containing $J_j \cap E$, we obtain a covering of E with nonoverlapping rectangles. Moreover there is a point of E on each side of each R_j, $s(R_j) \leqq s(S_1)/2 < 1/2^m$, $\sum s(R_j) \leqq \sum 4s(S_i)/2 < 8(H(E)+1)$ and finally $\left|\bigcup R_j\right| \leqq \sum s(R_j)^2 < 8(H(E)+1)/2^m$.

LEMMA 2. *Let $A(x,y)$ be linearly continuous on a square Q and let ε and δ be positive constants. Then if E is the set of points (x,y) of Q such that $|y'-y''| < \delta$ implies that $|A(x,y') - A(x,y'')| \leqq \varepsilon$, E is closed.*

Proof. Let $Q = [a,b] \times [c,d]$, and let F be the set of points x of $[a,b]$ such that $|y'-y''| < \delta$ implies that $|A(x,y') - A(x,y'')| \leqq \varepsilon$. If x' is not in F, there is a y' and a y'', with $|y'-y''| < \delta$ and $|A(x',y') - A(x',y'')| > \varepsilon$. Since $A(x,y')$ and $A(x,y'')$ are continuous, this inequality prevails in a neighborhood of x'. Thus, $[a,b] - F$ is open, F is closed, and $E = F \times [c,d]$ is closed.

LEMMA 3. *Suppose that R is the union of a finite number of rectangles, that $A(x,y)$ and $B(x,y)$ are bounded by M and linearly continuous on R, that $(B_x - A_y)$ is in $L_1(R)$, and that* (1) *holds in a neighborhood of each point of* $\operatorname{int}(R)$ *except perhaps the points of $E \cap \operatorname{int}(R)$. Then if E is closed and $H(E \cap \operatorname{int}(R)) \leqq \alpha$,*

$$\left| \int_{\partial R} A\,dx + B\,dy - \iint_R (B_x - A_y)\,dxdy \right| \leqq 4M\alpha.$$

Proof. We may assume that R is a square, and by Remark 1, we need only show that the inequality holds for every square contained in $\operatorname{int}(R)$. We may thus assume that E is a closed subset of R, and that $H(E) \leqq \alpha$.

Let $0 < \varepsilon < 1$ be given.

Since $(B_x - A_y)$ is in $L_1(R)$, there is an integer N such that $|C| < (\alpha+1)/N$ implies that $\iint_C |B_x - A_y|\,dxdy < \varepsilon$.

Since E is closed and $H(E) \leqq \alpha$, we can cover E with a finite number of open sets O_i such that $\delta(O_i) < 1/N$ and $\sum \delta(O_i) < (\alpha + \varepsilon)$. Then, for each O_i we can find a square Q_i, with $O_i \subset Q_i$ and $\delta(O_i) = s(Q_i)$. We then have $E \subset \bigcup Q_i$, $\sum s(Q_i) < (\alpha+\varepsilon)$ and $\left|\bigcup Q_i\right| < (\alpha+1)/N$.

It is clear that $T = R - \bigcup Q_i$ can be written as the union of a finite number of nonoverlapping rectangles none of which contain points of E. Applying Remark 2 to each of these rectangles we obtain

$$\int_{\partial T} A\,dx + B\,dy - \iint_T (B_x - A_y)\,dxdy = 0.$$

Since $\left|\bigcup Q_i\right| < (\alpha+1)/N$, we have

$$\left| \iint_{\cup Q_i} (B_x - A_y)\,dxdy \right| < \varepsilon,$$

and since $A(x, y)$ and $B(x, y)$ are bounded by M we also have

$$\left| \int_{\partial(\cup Q_i)} A\,dx + B\,dy \right| \leqq 4M \sum s(Q_i) < 4M(\alpha + \varepsilon).$$

By combining the last three inequalities we obtain

$$\left| \int_{\partial R} A\,dx + B\,dy - \iint_R (B_x - A_y)\,dxdy \right| < 4M(\alpha + \varepsilon) + \varepsilon.$$

Since ε was arbitrary this completes the proof.

3. Theorem. *Let $A(x, y)$ and $B(x, y)$ be bounded on a square Q and linearly continuous at each point of $Q - D$, where D is closed. Let $(B_x - A_y)$ be in $L_1(Q)$, and let the partial derivatives of $A(x, y)$ and $B(x, y)$ be finite at each point of $Q - \bigcup E_n$, where each E_n is closed. Then, if $H(D) = 0$ and $H(E_n) < \infty$ for each n,*

$$\int_{\partial Q} A\,dx + B\,dy = \iint_Q (B_x - A_y)\,dxdy.$$

Proof. Assume, first, that D is empty.

By Remark 1, we can assume that the conditions of the theorem are met in an open set containing Q, and if we let a point (x, y) of Q be in J if (1) does not hold in any neighborhood of (x, y), J is a closed set and, by Remark 2, we need only show that J is empty.

For each positive integer N, let (x, y) be in F_N if

$$|A(x + h, y) - A(x, y)| \leqq N|h|,$$
$$|A(x, y + k) - A(x, y)| \leqq N|k|,$$
$$|B(x + h, y) - B(x, y)| \leqq N|h|,$$
$$|B(x, y + k) - B(x, y)| \leqq N|k|,$$

whenever $|h| < 1/N$, $|k| < 1/N$, and all the quantities are defined. The sets F_N are closed [8, p. 80], and Q, with the exception of the sets E_n, is covered by $\bigcup F_N$. We may, therefore, write J as the union of the sets $J \cap F_N$ and the sets $J \cap E_n$. If J is not empty, it follows from the Baire Category Theorem that there is an open square I' such that $I' \cap J$ is not empty and either (a) $F_N \cap I' \supset J \cap I'$ for some N or (b) $E_n \cap I' \supset J \cap I'$ for some n.

The arguments given by Cohen [2] can be carried over to linearly continuous functions to show that (a) is impossible.

In order to show that (b) cannot occur, let I be any closed square contained in I'. We will show that (1) holds for I.

Let $\varepsilon > 0$ be given.

Let a point (x,y) of I be in G_N if whenever (x,y') and (x,y'') are in I and $|y'-y''| < 1/2^N$ then $|A(x,y')-A(x,y'')| \leqq \varepsilon$ and $|B(x,y')-B(x,y'')| \leqq \varepsilon$, and whenever (x',y) and (x'',y) are in I with $|x'-x''| < 1/2^N$ then $|A(x',y)-A(x'',y)| \leqq \varepsilon$ and $|B(x',y)-B(x'',y)| \leqq \varepsilon$. By Lemma 2, the sets G_N are closed, and it is clear that I is covered by $\bigcup G_N$. It is also clear that $G_1 \subset G_2 \subset G_3 \subset \cdots$. We can thus conclude that the sets $G_N \cap E_n$ are closed, that $I \cap E_n \supset G_N \cap E_n$, and finally that $H(I \cap E_n) = \lim_N H(E_n \cap G_N)$.

Thus, if we choose M sufficiently large, we have $H((I \cap E_n) - G_M) < \varepsilon$, and since $(B_x - A_y)$ is in $L_1(Q)$ we can also choose M large enough so that $\int\int_C |B_x - A_y|\, dxdy < \varepsilon$ whenever $|C| < 8(H(E_n \cap I)+1)/2^M$.

Applying Lemma 1, we can find a finite number of nonoverlapping rectangles R_j which cover $E_n \cap G_M$, and which satisfy

$$s(R_j) < 1/2^M,$$

$$\sum_s (R_j) < 8(H(E_n \cap G_M)+1),$$

$$\left|\bigcup R_j\right| < 8(H(E_n \cap G_M)+1)/2^M \;<\; 8(H(E_n \cap I)+1)/2^M$$

and such that each R_j has a point of G_M on each of its sides.

Let $R_j = [a,b] \times [c,d]$, and let (x',d) and (x'',c) be points of G_M which occur on the sides of R_j parallel to the x-axis. Since $s(R_j) < 1/2^M$, we have

$$\begin{aligned}|A(x,d)-A(x,c)| \leqq\; & |A(x,d)-A(x',d)| + |A(x',d)-A(x'',d)| \\ & + |A(x'',d)-A(x'',c)| + |A(x'',c)-A(x,c)| \leqq 4\varepsilon.\end{aligned}$$

This implies that

$$\int_a^b |A(x,c)-A(x,d)|\,dx \leqq (b-a)4\varepsilon \leqq 4\varepsilon s(R_j).$$

By combining this with a similar inequality for $B(x,y)$ we obtain

$$\left|\int_{\partial R_j} A\,dx + B\,dy\right| \leqq 8s(R_j)\varepsilon$$

and finally

$$\left|\int_{\partial(\cup R_j)} A\,dx + B\,dy\right| \leqq 8\sum s(R_j)\varepsilon \leqq 64(H(E_n)+1)\varepsilon. \tag{2}$$

Since $\left|\bigcup R_j\right| < 8(H(E_n)+1)/2^M$, we also have

$$\left|\iint_{\cup\partial R_j} (B_x - A_y)\,dxdy\right| < \varepsilon. \tag{3}$$

The interior of the set $T = I - \bigcup R_j$ can be written as the union of a finite number of open rectangles, and since (1) holds in a neighborhood of each point of int(T) except perhaps the points of E_n, and since

$$H(\operatorname{int}(T) \cap E_n) \leqq H((I \cap E_n) - G_M) \leqq \varepsilon,$$

we can apply Lemma 3 to $\operatorname{cl}(T)$ to obtain

$$\left| \int_{\partial T} A\,dx + B\,dy \;-\; \iint_T (B_x - A_y)\,dxdy \right| \leqq 4K\varepsilon, \tag{4}$$

where K is the bound of $A(x,y)$ and $B(x,y)$.

By combining (2), (3), and (4), we obtain

$$\left| \int_{\partial I} A\,dx + B\,dy - \iint_I (B_x - A_y)\,dxdy \right| < (64(H(E_n) + 1) + 1 + 4K)\varepsilon.$$

Since ε was arbitrary this implies that (1) holds for I, which in turn implies that (b) is impossible. We thus conclude that J is empty, and hence that (1) holds for Q if D is empty.

Suppose D is not empty, and that $1 > \varepsilon > 0$ is arbitrary.

Since $(B_x - A_y)$ is in $L_1(Q)$, if N is sufficiently large,

$$\left| \iint_C (B_x - A_y)\,dxdy \right| < \varepsilon,$$

whenever $|C| < 1/N$

Since $H(D) = 0$, and D is closed, we can cover D with a finite number of open squares R_j, with $s(R_j) < 1/N$, $\sum s(R_j) < \varepsilon$, and $\left|\bigcup R_j\right| < \varepsilon/N$.

We thus have

$$\left| \iint_{\cup R_j} (B_x - A_y)\,dxdy \right| < \varepsilon,$$

and

$$\sum \left| \int_{\partial R_j} A\,dx + B\,dy \right| \leqq 4K \sum s(R_j) < 4K\varepsilon,$$

where K is the bound on $A(x,y)$ and $B(x,y)$.

The set $T = Q - \bigcup R_j$ can be written as the union of a finite number of non-overlapping rectangles to which the first part of the proof applies. Thus

$$\int_{\partial T} A\,dx + B\,dy \;-\; \iint_T (B_x - A_y)\,dxdy = 0.$$

By combining the last three inequalities, we obtain

$$\left| \int_{\partial Q} A\,dx + B\,dy \;-\; \iint_Q (B_x - A_y)\,dxdy \right| < (4K + 1)\varepsilon.$$

Since ε was arbitrary, (1) holds for Q and the proof is complete.

4. **Example.** In this section we will give an example which shows that the conditions given in the theorem can not be weakened substantially. The example given by Maker [4], can be used to show that if B_x or A_y fails to exist on a closed set which has Hausdorff dimension one but which is not σ-finite then the equality (1) may fail. By modifying the example given by Tolstoff [7], we will show that even when B_x and A_y exist everywhere, some assumption must be made on the partial derivatives B_y and A_x which do not appear in (1), and that the assumption $H(1+\varepsilon, E) = 0$, on the set E where they fail to exist, is not sufficient. We will use the same example to show that the condition $H(D) = 0$ on the set of points where the functions involved are not linearly continuous can not be weakened. (In particular, it is not sufficient to assume that $A(x, y)$ and $B(x, y)$ are continuous functions of x for almost every y and continuous functions of y for almost every x.)

Let

$$g(x, a, b) = 1 + \cos 2\pi \left[\frac{x-a}{b-a} - \frac{1}{2}\right], \quad a \leqq x \leqq b,$$

$$g(x, a, b) = 0 \qquad \text{otherwise,}$$

and let

$$h(x, y; (a, b) \times (c, d)) = g(x, a, b)\, g(y, c, d).$$

Let C be a Cantor perfect set, and call each of the intervals removed in the nth step of the construction of C a Δ_n.

Define $f(x, y)$ on $Q = [0,1] \times [0,1]$ by

$$\begin{aligned} f(x, y) &= 4h(x, y; \Delta_n \times \Delta_n)/4^n|\Delta_n|^2 && \text{on } \Delta_n \times \Delta_n \\ &= -8h(x, y; \Delta_n \times \Delta_{n-1})/4^n|\Delta_n||\Delta_{n-1}| && \text{on } \Delta_n \times \Delta_{n-1} \\ &= 0 && \text{otherwise.} \end{aligned}$$

Finally, define $A(x, y)$ and $B(x, y)$ on Q by

$$A(x, y) = \int_0^y f(x, u)\, du \quad \text{and} \quad B(x, y) = \int_0^x f(u, y)\, du.$$

It is clear that B_x and A_y exist and are equal to $f(x, y)$ everywhere, so that $(B_x - A_y) = 0$.

We can notice that if (x, y) is in a rectangle $\Delta_n \times [0,1]$ or a rectangle $[0,1] \times \Delta_n$. $A(x, y)$ is continuous and A_x exists and is finite. A similar observation can be made for $B(x, y)$ and B_y.

If we let M_n denote the maximum of the absolute value of $A(x, y)$ on a rectangle $\Delta_n \times [0,1]$, a close examination of $A(x, y)$ shows that $M_n = 8/4^n|\Delta_n|$. Since $A(x, y) = 0$ on $C \times [0,1]$, $A(x, y)$ will be continuous at points of $C \times C$ (and hence on Q) provided $\text{Lim}_n M_n = 0$, which will be the case if $|\Delta_n| = 1/r^n$ with $r < 4$. On the other hand if $r = 4$, we will have $\text{Lim}_n M_n = 8$ so that $A(x, y)$ will be bounded. A similar observation can be made for $B(x, y)$. In the first case, we can choose $r < 4$ in such a way that $H(1 + \varepsilon, C \times C) = 0$. If $r = 4$, we have $\infty > H(C \times C) > 0$.

In order to complete our discussion, we must show that $\int_{\partial Q} A\,dx + B\,dy \neq 0$. If we notice that

$$\int_{\Delta_n} \int_0^1 f(x,y)\,dxdy = 0, \qquad n \geqq 1,$$

$$\int_{\Delta_n} \int_0^1 f(x,y)\,dydx = 1, \qquad n = 1,$$

$$\int_{\Delta_n} \int_0^1 f(x,y)\,dydx = 0, \qquad n \geqq 2,$$

we see that

$$\int_{\partial Q} B\,dy = \int_0^1 B(1,y)\,dy = \int_0^1 \int_0^1 f(x,y)\,dxdy$$

$$= \sum 2^{n-1} \int_{\Delta_n} \int_0^1 f(x,y)\,dxdy = 0,$$

while

$$\int_{\partial \Phi} A\,dx = -\int_0^1 A(x,1)\,dx = -\int_0^1 \int_0^1 f(x,y)\,dydx$$

$$= -\sum 2^{n-1} \int_{\Delta_n} \int_0^1 f(x,y)\,dydx = -1.$$

5. **Comments.** As a corollary to the theorem presented here, one can easily deduce an extension of the Looman-Menchoff Theorem [5, p. 199] which includes the extensions given by Maker [4] and Tolstoff [8].

It should be noted also that the proof does not extend to higher dimensions, at least not for linearly continuous functions. (However, if one modifies the continuity conditions suitably, the proof can be carried over.)

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