Course MM1005

Limits: Formal definition

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Motivation

Working with statistics, you will have to work and master several notions of convergence of sequences of random variables (functions) on a probability space over U, $\{X_n\}_n$:

1. Almost surely: $X_n \stackrel{a,s,}{\to} X$ as $n \to +\infty$.
2. In distribution/law: $X_n \stackrel{d}{\to} X$ as $n \to +\infty$.
3. In probability: $X_n \stackrel{\mathbb{P}}{\to} X$ as $n \to +\infty$.

To be able to grasp those ideas we need to understand the idea of **pointwise convergece** first.

Pointwise convergence

If we have continuous time, say that the family of random variables is $\{X_t\}_{t\geq 0}$, we consider the function $t \mapsto X_t(\omega)$.

Definition

We say that X_t converges pointwise to X if **for all** $u \in U$,

$$\lim_{t\to+\infty} X_t(u) = X(u),$$

That is, for all $\varepsilon > 0$, there exists $\omega > 0$ such that for all $t \geq \omega$ $|X_t(u) - X(u)| < \varepsilon$.

That may be a huge step right now to grasp, so let us forget about u, and focus on the expression as a function of t, and try to understand the notion of limit.

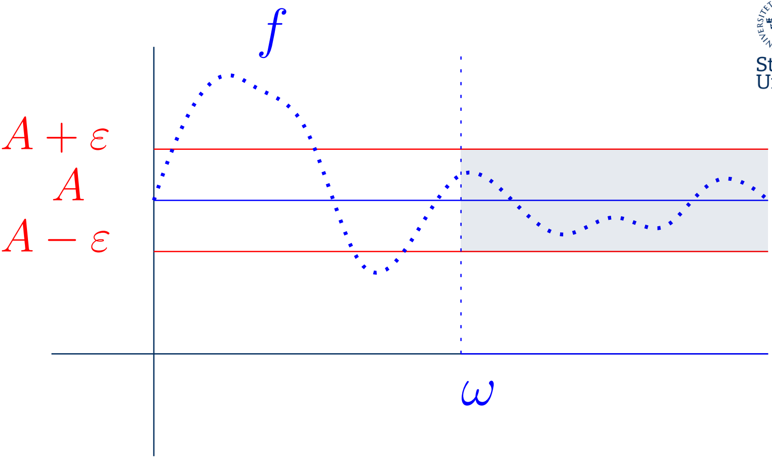
Limit towards $+\infty$

Definition

Let $f : (0, +\infty) \to \mathbb{R}$ be a given function. Let $A \in \mathbb{R}$. We say that f converges to A as t goes to infinity, and we denote it by

$$\lim_{t \to +\infty} f(t) = A,$$

if, and only if, for all arbitrarily small $\varepsilon > 0$,there exists $\omega > 0$ such that, for all $t \geq \omega$ we have that $|f(t) - A| < \varepsilon$.



Intuitively: Tell me how far away you would accept the values of your function to be from A (This is the role of ε).
So, I can find a value large enough (This is the role of ω), such that from that point onward, all the values of f are as closed to A as we have prescribed.

Example

Lets show that $\lim_{t \to +\infty} \frac{1}{t} = 0$.
Given $\varepsilon > 0$, we need to find ω such that for all $t \geq \omega$

$$\left|\frac{1}{t} - 0\right| = \frac{1}{t} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < t,$$

So, fixed ε we can take $\omega = \frac{2}{\varepsilon}$.

Example

Lets show from the definition that

$$\lim_{t \to +\infty} \frac{t-1}{t} = 1,$$

Given ε, we want to find ω such that for all $t \geq \omega$ we have that

$$\left| \frac{t-1}{t} - 1 \right| < \varepsilon \Leftrightarrow \frac{1}{t} < \varepsilon,$$

So, as before, it is enough to take $\omega = \frac{2}{\varepsilon}$.

Complete the slide

Lets show from the definition that

$$\lim_{t\to+\infty} \frac{t-1}{2t} = \frac{1}{2},$$

Given ε, we want to find ω such that for all $n \geq \omega$ we have that

$$\_\_\_\_\_\_\_\_\_\_ < \varepsilon \Leftrightarrow \_\_\_\_\_\_\_\_\_\_ < \varepsilon,$$

So, it is enough to take $\omega =$ __________.

Exercise

Calculate the following limit and show from the definition that your answer is correct:

$$\lim_{x \to +\infty} \frac{1 - x^2}{1 + x^2},$$

Practice Exercises

Prove, using the definition of limit as $x \to +\infty$, that

- $\lim\limits_{x \to +\infty} e^{-x} = 0,$
- $\lim\limits_{x \to +\infty} e^{-x^2} = 0,$
- $\lim\limits_{x \to +\infty} \dfrac{3x+5}{x} = 3,$
- $\lim\limits_{x \to +\infty} \left(\sqrt{x^2+1} - x\right) = 0,$
- $\lim\limits_{x \to +\infty} \dfrac{x+1}{2x-3} = \frac{1}{2},$

Thank you for your attention!