4.32 The Cauchy-Goursat Theorem

For the sake of completeness, we shall give here a proof of the Cauchy-Goursat Theorem of simply connected domains. The argument here uses that we know that the theorem holds for triangles, and so it also does for squares (see Remark 4.27 above). We start by a technical lemma:

Lemma 4.33. *Let f be holomorphic on and inside a simple, piecewise-smooth, closed curve γ. For every $\varepsilon > 0$, the closed region $\overline{\mathrm{Int}(\gamma)} = \mathrm{Int}(\gamma) \cup \{\gamma\}$ can be covered by a finite number N of closed squares Q_j, $j = 1, 2, \ldots, N$, such that for each of these, the following property holds:*

$$\exists z_j \in Q_j \cap \overline{\mathrm{Int}(\gamma)} \quad \textit{such that} \quad \left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon, \quad \forall z \in Q_j \cap \overline{\mathrm{Int}(\gamma)} \setminus \{z_j\}. \tag{4.7}$$

Proof. Suppose, by contradiction, that the statement is false. Then there exists some $\varepsilon > 0$ such that for every finite covering, there is at least one square where (4.7) fails.

Start with an initial covering of the region with N_0 squares $Q_k^{(0)}$, $k = 1, 2, \ldots, N_0$, each of side length d, and repeatedly refine this covering by subdividing each square into four smaller squares of side $d/2$, $d/4$, and so on. At the n-th step, the covering consists of N_n squares $Q_k^{(n)}$, $k = 1, 2, \ldots, N_n$, each of side $d/2^n$.

By assumption, for each subdivision level n, inequality (4.7) fails in at least one of the $Q_k^{(n)}$. Let

$$\mathcal{F}_n := \{k : (4.7) \text{ fails}\}.$$

Let

$$A_n = \bigcup_{k \in \mathcal{F}_n} Q_k^{(n)} \cap \overline{\mathrm{Int}(\gamma)}, \quad n \in \mathbb{N}_0.$$

Clearly, $A_n \supset A_{n+1}$, since if (4.7) fails for a square of level $n+1$, it also fails for the level-n square containing it.

Since by hypothesis $A_n \neq \emptyset$ for all n, choose a point $w_n \in A_n$, and consider the sequence $\{w_n\}_{n=0}^{\infty}$. As $w_n \in \overline{\mathrm{Int}(\gamma)}$ and $\overline{\mathrm{Int}(\gamma)}$ is a closed and bounded subset of $\mathbb{C}$ (hence compact), by Bolzano-Weierstrass theorem (see Remark 1.18), there exists a convergent subsequence $\{w_{n_j}\}_j$ with limit $w \in \overline{\mathrm{Int}(\gamma)}$.

Since $f(z)$ is holomorphic on $\overline{\mathrm{Int}(\gamma)}$, and hence at w, there exists $\delta(\varepsilon) > 0$ such that

$$\left| \frac{f(z) - f(w)}{z - w} - f'(w) \right| < \varepsilon, \quad \forall z \in D(w, \delta) \setminus \{w\}.$$

Now consider a covering at level n_0, with n_0 large enough, such that $\sqrt{2} d/2^{n_0} < \delta$. Since w is in each A_n (as A_n is closed and all w_{n_j} eventually lie in A_n), we have that $w \in Q_{k_0}^{(n_0)} \cap \overline{\mathrm{Int}(\gamma)} \subset A_{n_0}$ for some k_0.

We then arrive at the contradiction, since we have that

$$\exists w \in Q_{k_0}^{(n_0)} \cap \overline{\mathrm{Int}(\gamma)} \quad \text{such that} \quad \left| \frac{f(z) - f(w)}{z - w} - f'(w) \right| < \varepsilon, \quad \forall z \in Q_{k_0}^{(n_0)} \cap \overline{\mathrm{Int}(\gamma)} \setminus \{w\}.$$

which contradicts the assumption that (4.7) fails in that square. □

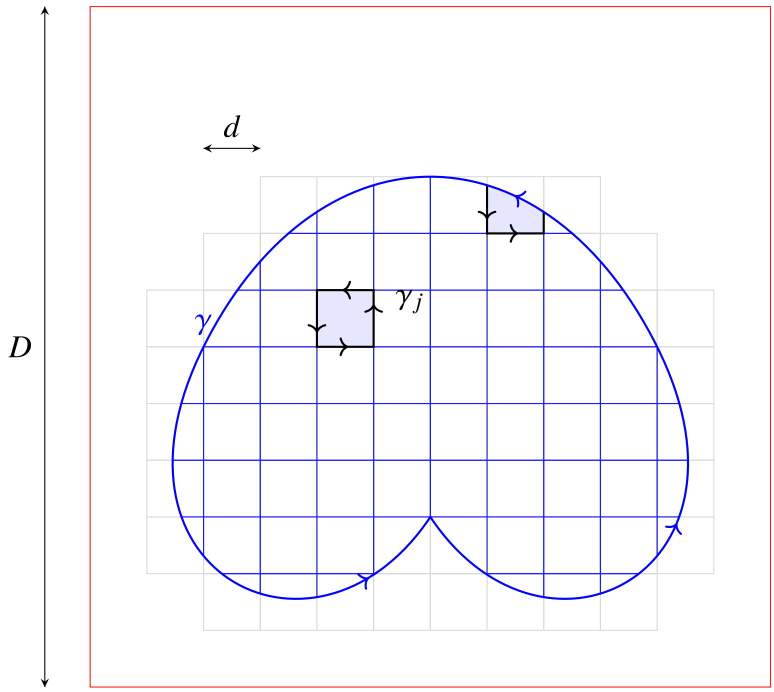

Figure 4.6: $\int_\gamma f(z)\mathrm{d}z = \sum_j \int_{\gamma_j} f(z)\mathrm{d}z$. Besides, some of the squares Q_j lie inside the domain, while others don't.

Theorem 4.34: Cauchy-Goursat for simple curves

Let $f \in \mathcal{O}(\overline{\mathrm{Int}(\gamma)})$, where Γ is a piecewise-smooth simple closed curve. Then

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Proof. By virtue of the previous lemma, for all $\varepsilon > 0$ it is possible to find a finite covering of $\mathrm{Int}(\gamma)$, the closed region bounded by γ, with N closed squares Q_j, $j = 1, 2, \ldots, N$, such that for each square there exists a point $z_j \in Q_j \cap \overline{\mathrm{Int}(\gamma)}$ and a function $\delta_j(z)$ defined by

$$\delta_j(z) = \begin{cases} \dfrac{f(z) - f(z_j)}{z - z_j} - f'(z_j), & z \neq z_j \\ 0, & z = z_j \end{cases}$$

which is continuous and satisfies $|\delta_j(z)| < \varepsilon$ for all $z \in Q_j \cap \overline{\mathrm{Int}(\gamma)}$.

Let γ_j, $j = 1, 2, \ldots, N$, be the positively oriented boundary path of $Q_j \cap \overline{\mathrm{Int}(\gamma)}$. Using the identity

$$f(z) = f(z_j) + f'(z_j)(z - z_j) + \delta_j(z)(z - z_j),$$

we obtain, for each γ_j:

$$\int_{\gamma_j} f(z)\mathrm{d}z = (f(z_j) - z_j f'(z_j)) \int_{\gamma_j} \mathrm{d}z + f'(z_j) \int_{\gamma_j} z\mathrm{d}z + \int_{\gamma_j} \delta_j(z)(z - z_j)\mathrm{d}z.$$

Since $\int_{\gamma_j} \mathrm{d}z = 0$ and $\int_{\gamma_j} z\mathrm{d}z = 0$ (as their integrands have primitives z and $z^2/2$, respectively, and γ_j is closed), it follows that

$$\int_{\gamma_j} f(z)\mathrm{d}z = \int_{\gamma_j} \delta_j(z)(z - z_j)\mathrm{d}z.$$

Summing over all $j = 1, \dots, N$, we have

$$\sum_{j=1}^{N} \int_{\gamma_j} f(z)\mathrm{d}z = \int_{\gamma} f(z)\mathrm{d}z,$$

and therefore, using the triangle inequalities and Theorem 4.10, we obtain that

$$\left|\int_{\gamma} f(z)\mathrm{d}z\right| = \left|\sum_{j=1}^{N} \int_{\gamma_j} \delta_j(z)(z - z_j)\mathrm{d}z\right| \le \sum_{j=1}^{N} \int_{\gamma_j} \left|\delta_j(z)(z - z_j)\right| \mathrm{d}z$$

Notice that on γ_j, we have that

$$\left|\delta_j(z)(z - z_j)\right| \le \varepsilon\sqrt{2}d,$$

where d is the side-length of the squares Q_j, and so $\sqrt{2}d$ is its diameter. Thus

$$\int_{\gamma_j} \left|\delta_j(z)(z - z_j)\right| \mathrm{d}z \le \sqrt{2}d\,\ell\left(\gamma_j\right)\varepsilon.$$

Notice that the length is equal to $4d$ if Q_j lies inside the domain, and if not,

$$\ell\left(\gamma_j\right) \le 4d + L_j,$$

where L_j is the length of $\gamma_j \cap \gamma$ (which we define as zero if the the intersection of the cube Q_j with γ is empty). Hence we have that

$$\left|\int_{\gamma} f(z)\mathrm{d}z\right| \le \sqrt{2}\varepsilon\left(4Nd^2 + d\sum_{j=1}^{N} L_j\right).$$

Notice that $L := \sum_{j=1}^{N} L_j$ is the length of γ.

By the Jordan Curve theorem (Theorem 3.22), $\mathrm{Int}(\gamma)$ is bounded. So we can find a square of finite side D such that it contains $\cup_{j=1}^{N} Q_j$, which has area Nd^2. So $Nd^2 \le D^2$. Moreover

$$dL \le DL.$$

So we have, in conclusion, that

$$\left|\int_{\gamma} f(z)\mathrm{d}z\right| < \varepsilon\sqrt{2}D(4D + L),$$

where the constant on the right hand side is independent on the covering $\{Q_j\}$, and since ε is arbitrary, the claim follows. □