Complex Analysis.

[Complex-differentiability & Holomorphy ($\mathcal{O}(\Omega)$)

$f = u + iv$] $\longrightarrow$ [Cauchy-Riemann equations]

- $f \in \mathcal{O}(\Omega) \Rightarrow \left[\frac{\partial f}{\partial \bar{z}}(z) = 0 \quad \forall z \in \Omega.\right]$

$$\Leftrightarrow \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y} \quad \forall z \in \Omega$$

$$\Leftrightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x} \end{cases} \quad \forall z \in \Omega$$

CR + $u, v \in \mathcal{C}^1(\Omega) \Rightarrow f \in \mathcal{O}(\Omega)$

Examples

- Polynomials, rational functions
  exponential, trigonometrical functions
  Logarithm, powers
  [Branches of multi-valued functions]

Integration along contours & properties.

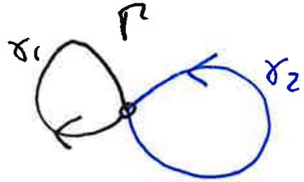
$$\int_\Gamma f(z)\,dz = \sum_{j=1}^{n} \int_a^b f(\gamma_j(t))\,\dot{\gamma}_j(t)\,dt.$$

$\gamma_j : [a,b] \to \mathbb{C} \qquad \Gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$

[$\gamma_j \equiv$ simple curves

Thm

$$\left|\int_\Gamma f(z)\,dz\right| \leq \sup_{z \in \Gamma} |f(z)| \cdot \ell(\Gamma)$$



[$\Gamma = \gamma_1 + \gamma_2$]

$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

Thm Path independence if a primitive exists.

$\iff \forall \gamma \subset \Omega$ closed contour $\int_\gamma f(z)\,dz = 0$.

Thm Cauchy's theorem (on a triangle): $\boxed{f \in \mathcal{O}(T) \Rightarrow \int_{\partial T} f(z)\,dz = 0}$

T triangle.

Thm Cauchy-Goursat theorem

If $f \in \mathcal{O}(\Omega)$ & Ω is simply connected domain.

$\Rightarrow \int_\gamma f(z)\,dz = 0 \quad \forall \gamma \subset \Omega$ closed contour.

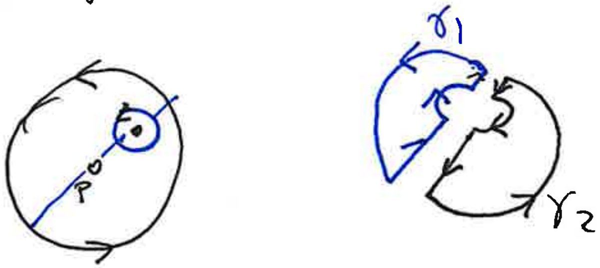
$\Rightarrow \exists F \in \mathcal{O}(\Omega)$ s.t. $F' = f$. on Ω.

[We only proved this for star-shaped domains]

Theorem Cauchy's integral formula

$f \in \mathcal{O}(\overline{D}(p,r))$ $\quad z \in D(p,r) \Rightarrow f(z) = \frac{1}{2\pi i}\int_{\partial D(p,r)} \frac{f(w)}{w-z}\,dw$

"Proof"



$$0 = \left(\int_{\gamma_1} + \int_{\gamma_2}\right)\left(\frac{f(w)}{w-z}\,dw\right.$$

$$= \left(\int_{\partial D(p,r)} - \int_{\partial D(z,\varepsilon)}\right)\frac{f(w)}{w-z}\,dw.$$

$\rightarrow$ Calculate "explicitly"

$$\int_{\partial D(z,\varepsilon)} \frac{f(w)}{w-z}\,dw \xrightarrow[\varepsilon \to 0^+]{} f(z)\,2\pi i$$

Cauchy's integral formula

$f \in \mathcal{O}(\Omega)$, $\Gamma \subset \Omega$ closed simple contour $z \in \mathrm{Int}(\Gamma)$ → Fundamental Theorem of Algebra.

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,dw.$$

Techical Lemma

$h \in \mathcal{C}(\Gamma)$, Γ closed simple contour.

$$g_n(z) = \int_\Gamma \frac{h(w)}{(w-z)^n}\,dw \in \mathcal{O}(\mathbb{C}\setminus\Gamma)$$

Liouville's theorem

$f \in \mathcal{O}(\mathbb{C}) \cap \mathcal{B}(\mathbb{C}) \Rightarrow f$ constant

Generalised Cauchy's integral Formula.

$f \in \mathcal{O}(\Omega) \Rightarrow$ i) $f \in \mathcal{C}^\infty(\Omega)$ & $f^{(n)} \in \mathcal{O}(\Omega)$

ii) $f^{(n)}(z) = \dfrac{n!}{2\pi i}\displaystyle\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\,dw \quad z \in \mathrm{Int}(\Gamma)$

+ Path indep.

Morera's theorem

$f \in \mathcal{C}(\Omega)$: $\int_\Gamma f(z)\,dz = 0$ $\forall \Gamma \subset \Omega$ (c.s.c.) $\Rightarrow f \in \mathcal{O}(\Omega)$

If $\Gamma = \partial D(z, R)$ (Estimates on derivatives)

$$|f^{(n)}(z)| \le \frac{n!}{2\pi\, R^n}\,\|f\|_{C_R}$$

Power series Expansions of holomorphic functions on $D(z_0, r)$

Series Representations.

Power series $\left[\sum_{n\geq 0} a_n (z-z_0)^n\right]$ $a_n \in \mathbb{C}, n \geq 0$.

Let $\alpha := \inf_{n\geq 1}\left(\sup_{k\geq n} |a_k|^{1/k}\right) =: \limsup |a_k|^{1/k} \in [0, +\infty)$.

$\rho := \frac{1}{\alpha} \in (0, +\infty]$. (Radius of convergence)

Theorem (Cauchy-Hadamard)

If $|z-z_0| < \rho$ the series converges absolutely &

$\forall r < \rho$ the series converges uniformly* on $\overline{D}(z_0, r)$

If $|z-z_0| > \rho$ the series diverges

If $F_N(z) := \sum_{n=0}^{N} a_n (z-z_0)^n$ & $F(z) := \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

F_N converges uniformly to F on $K \iff \lim_{N\to+\infty} \|F_N - F\|_K = 0$

where $\|F_N - F\|_K = \sup_{z\in K} |F_N(z) - F(z)|$ $\left(\|F_N - F\|_\infty \right)$

Weierstrass M-test.

If $f_n : T \to \mathbb{C}$, $n \geq 0$

$\{M_n\}_n \subset \mathbb{R}_+$ s.t. $\forall n$ $\forall z \in T$ $|f_n(z)| \leq M_n$.

$\sum\limits_{n\geq 0}' M_n < +\infty$

$\Rightarrow \sum\limits_{n=0}^{\infty}' f_n(z)$ converges uniformly on T

Corollary The geometric series $\sum\limits_{n\geq 0} z^n$ converges uniformly on $\overline{D(0,r)}$ for all $0<r<1$.

Theorem

i) If $f_n \in \mathcal{C}(K)$, $n \geq 0$ and f_n converges uniformly to $f \Rightarrow f \in \mathcal{C}(K)$

ii) If $f_n \in \mathcal{C}(\Omega)$ and on any compact subset of Ω,
$n \geq 0$

f_n converges uniformly to f then $\lim\limits_n \int_\Gamma f_n(z)dz = \int_\Gamma f(z)dz$

for all contour $\Gamma \subset \Omega$.

iii) $f_n \in \mathcal{O}(\Omega)$ and f_n converging uniformly to f on compact subsets of $\Omega \Rightarrow f \in \mathcal{O}(\Omega)$

Ω s.c. (or Morera's theorem)

$$\frac{1}{z-w} = \frac{1}{z-z_0+(w-z_0)} = \frac{1}{z-z_0}\left(\frac{1}{1-\frac{w-z_0}{z-z_0}}\right) \qquad \left|\frac{w-z_0}{z-z_0}\right| < s < 1. \ (\#)$$

$$= \frac{1}{z-z_0}\sum_{n\geq 0}\left(\frac{w-z_0}{z-z_0}\right)^n$$ uniformly in w, z s.t. $(\#)$

Taylor's theorem

- $f \in \mathcal{O}(D(z_0, r)) \Rightarrow f(w) = \sum_{n\geq 0} a_n (w-z_0)^n \quad w \in D(z_0, r)$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{\partial(z_0,s)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad 0<s<r.$$

Theorem

- If $f(w) = \sum_{n\geq 0} a_n (w-z_0)^n$, it converges to a holomorphic function on $D(z_0, \rho)$ $\rho \equiv$ radius of convergence of the series

$$= \frac{1}{\limsup |a_n|^{1/n}}$$

Moreover $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

$$\left\{\begin{array}{c}\text{Holomorphic}\\ \text{functions}\\ \text{on } D(z_0,r)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Analytic}\\ \text{function}\\ \text{on } D(z_0,r)\end{array}\right\}.$$

Corollary

If $\sum a_n (z-z_0)^n = f(z)$ on $D(z_0,r)$

$$\Rightarrow \quad f'(z) = \sum_{n\geq 1} n a_n (z-z_0)^{n-1}$$

$\mathcal{A}(z_0, 0, R) =: D^*(z_0, R)$

Laurent Series.

$f \in \mathcal{O}(\mathcal{A}(r, R))$ $\quad \mathcal{A}(z_0, r, R) = \{z : r < |z - z_0| < R\}$.

$$\Rightarrow f(z) = \sum_{j \in \mathbb{Z}} a_j (z - z_0)^j$$

with uniform convergence on $\mathcal{A}(z_0, \rho_1, \rho_2)$; $0 \le r < \rho_1 < \rho_2 < R$

and $$a_j = \frac{1}{2\pi i} \int_{C(z_0, s)} \frac{f(w)}{(w - z_0)^{j+1}} \, dw \quad j \in \mathbb{Z}.$$

$f \in \mathcal{O}(D^*(z_0, R))$ & $f(z) = \sum_{j \in \mathbb{Z}} a_j (z - z_0)^j$ it's Laurent expansion

i) If $a_j = 0$ $j < 0$, z_0 is a removable singularity

ii) If $a_j = 0$ for some $j < -m$, $a_{-m} \neq 0$, z_0 is a pole of order m

iii) otherwise, it's an essential singularity.

Riemann's removable singularity.

$f \in (\mathcal{O} \cap \mathcal{B})(D(z_0, r) \setminus \{z_0\}) \Rightarrow f$ has a removable singularity.
so it can be redefined at z_0 so $f \in \mathcal{O}(D(z_0, r))$

If f has a pole of order m at $z=z_0$ (on $D^*(z_0, r)$)

$(z-z_0)^m f(z)$ has a removable singularity.

& abusing notation

$(z-z_0)^m f(z) \in \mathcal{O}(D(z_0, r))$

So

$$(z-z_0)^m f(z) = (z-z_0)^m \sum_{j=-m}^{\infty} a_j (z-z_0)^j$$

$$= \sum_{j=-m}^{\infty} a_j (z-z_0)^{j+m} = \sum_{j=0}^{\infty} a_{j-m} (z-z_0)^j$$

$a_{-1} \equiv$ The residue of f at z_0: $\operatorname{Res}(f, z_0) = a_{-1}$

How to calculate it? (Ass. $z_0 = 0$

$$z^m f(z) = a_{-m} + a_{-m+1} z + \dots + a_{-1} z^{m-1} + a_0 z^m + \dots$$

$$\lim_{z \to 0} \left(\frac{d}{dz} (z^m f(z)) \right) = a_{-m+1} \qquad \lim_{z \to 0} \frac{d^2}{dz^2} (z^m f(z)) = 2a_{-m+2}$$

$$\left[\lim_{z \to 0} \frac{d^{m-1}}{dz^{m-1}} (z^m f(z)) = (m-1)!\, a_{-1}. \right]$$

So if f has a pole of order 2 at zero.

$$f(z) = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + \dots \qquad \text{for } |z| < r.$$

$$\Rightarrow \quad z^2 f(z) = a_{-2} + a_{-1} z + a_0 z^2 + a_1 z^3 + \dots$$

$$\Rightarrow \quad \frac{d}{dz}\left(z^2 f(z)\right) = a_{-1} + 2a_0 z + 3a_1 z^2 + \dots \xrightarrow[z \to 0]{} a_{-1}.$$

$$\boxed{\operatorname{Res}(f, 0) = \lim_{z \to 0} \frac{d}{dz}\left(z^2 f(z)\right)}$$

Theorem (Residue theorem)

If f is meromorphic in Ω and Γ is a simple closed contour $\Gamma \subset \Omega$, $z_1, \ldots, z_k \in \mathrm{Int}(\Gamma)$ poles

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$$

Applications to Real integrals

The Argument principle

- If f is meromorphic inside Γ
- Γ simple closed contour

$f \in \mathcal{O}(\Gamma)$

$$\int_\Gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i \left(N_0(f) - N_p(f) \right)$$

Rouché's theorem

$f, g \in \mathcal{O}(\overline{\mathrm{Int}(\Gamma)})$

$|f(z) - g(z)| < |f(z)| \quad \forall z \in \Gamma.$

$\Rightarrow N_0(f) = N_0(g)$ in $(\mathrm{Int}\,\Gamma)$

Idea:

$\forall z \in \Gamma \quad \Big|1 - \underbrace{\frac{g(z)}{f(z)}}_{F(z)}\Big| < 1 \Rightarrow F(\Gamma) \subset D(1,1)$

$0 = \int_\Gamma (\log F(z))'\,dz$

$\boxed{\text{Open mapping theorem}}$ (Consequence of Rouché's)

$f \in \mathcal{O}(\Omega)$ non-constant $\Rightarrow$ $f(U)$ is open $\forall U \subset \Omega$ open.

$\rightsquigarrow$ If $f(z_1) = w_1$, $f^{(1)}(z_1) = \dots = f^{(k-1)}(z_1) = 0$; $f^{(k)}(z_1) \neq 0$] locally, $f(z)$ behaves like $(z-z_1)^k + w_1$ (so every $w \neq w_1$ has k different pre-images)

Corollary If $f \in \mathcal{O}(\Omega)$ and f is one-to-one $\Rightarrow$ $f'(z) \neq 0$. $\forall z \in \Omega$.

Maximum principle.

$f \in \mathcal{O}(\Omega)$, non constant $\Rightarrow$ $|f|$ can't attain its maximum value in Ω.

Conformal maps

Prop. If $f: U \to V$, $f \in \mathcal{O}(U)$ & bijective $\Rightarrow$
(f is a conformal map)

- $f'(z) \neq 0 \ \forall z \in U$
- $f^{-1} \in \mathcal{O}(V)$ & $\left[(f^{-1})'(v) = \dfrac{1}{f'(f^{-1}(v))} \right]$

Def. U, V are conformally equivalent $\Leftrightarrow$ $\exists f: U \to V$ biholomorphic (conformal)

Examples of conformal maps

i) Translations. $z \mapsto z+b \quad b \in \mathbb{C}$

ii) Dilations/rotations: $z \mapsto az \quad a \in \mathbb{C}^*$

iii) Inversion: $z \mapsto 1/z$

$\boxed{\mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty}$

Möbius transformations.

$$z \longmapsto \frac{az+b}{cz+d} \qquad ad-bc \neq 0.$$

If $c \neq 0$

$$= \frac{\frac{a}{c}(cz+d) + b - \frac{ad}{c}}{cz+d}$$

$$= \frac{a}{c} + \frac{bc-ad}{c^2z+dc}$$

$$z \mapsto cz+d \longmapsto \frac{bc-ad}{c^2z+dc} \longmapsto \frac{az+b}{cz+d}.$$

If $c = 0$

$$z \longmapsto \frac{a}{d}z + \frac{b}{d}$$

Caley Map

$$C(z) = \frac{z-i}{z+i}$$

Lemma Mobius transformations map clines to cline.

The equation of a cline

$$A|z|^2 + B\,\mathrm{Re}(\bar{\gamma} z) + C = 0 \qquad \begin{bmatrix} A=0 & \text{line} \\ A\neq 0 & \text{circle} \end{bmatrix}$$

It's enough to prove it by [Translations; Dilations; Inversion.]

Let T be a Mobius transformation

i) T can be expressed by composition of translations, dilations and inversions

ii) $T: \mathbb{C}_\infty \to \mathbb{C}_\infty$

iii) T is conformal at every point except at the pole.

iv) The composition of Mobius transformations is a Mobius transformation.

v) Mobius transformations are invertible.

vi) T Map clines to clines

Cross-ratio $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$

$$[\underbrace{z_1\, z_2\, z_3\, z_4}_{\text{Fixed and different}}] = \frac{z_1 - z_2}{z_1 - z_4} \Big/ \frac{z_3 - z_2}{z_3 - z_4} \cong$$

$z_1 \mapsto T(z_1)$ Mobius transformation such that

$$\begin{bmatrix} T(z_2) = 0 \\ T(z_4) = \infty \\ T(z_3) = 1 \end{bmatrix}$$

Mobius transformations preserve cross-ratio

$$[z_1, z_2, z_3, z_4] = [T(z_1), T(z_2), T(z_3), T(z_4)].$$

Given z_2, z_3, z_4, if C is the cline going through $z_2 z_3 z_4$ (with that orientation) we say that $z \notin C$ belongs to the left (right) of C if

$$\operatorname{Im}[z, z_2, z_3, z_4] > 0 \quad (<)$$

Thm If T is a Mobius transformation and C is a cline oriented according to $z_2, z_3, z_4 \in C$, if z is to the left of C, so it's $T(z)$ w.r.t. $T(C)$ with the orientation given by $T(z_2), T(z_3), T(z_4)$

Harmonic functions

$u \in \mathcal{C}^2(\Omega)$ is harmonic $\iff$ $\Delta u = 0$. $\forall z \in \Omega$.

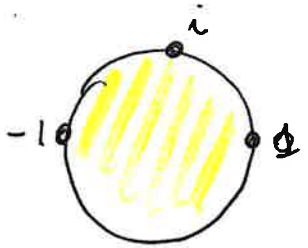
If $g \in \mathcal{O}(\Omega)$, $g : \Omega' \to \Omega$ conformal, then $u(g(z)) \in$ harmonic in Ω'

$$\Delta\big(u(g(z))\big) = (\Delta u)(g(z)) \cdot |g'(z)|^2$$

Theorem
$\forall u \in h(\Omega)$, Ω s.c. domain $\exists v \in h(\Omega)$: $u + iv \in \mathcal{O}(\Omega)$.

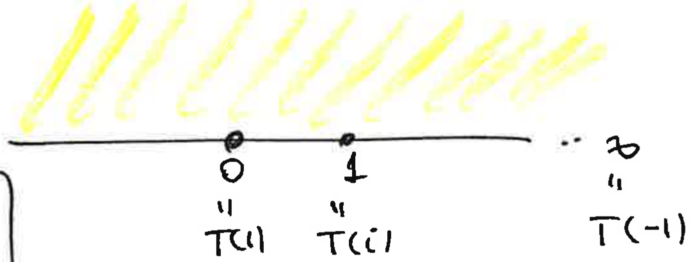
$\Omega = D(z_0, r)$

$$\begin{tikzcd} \mathcal{O}(D(z_0,r)) \arrow[r, "\mathrm{Re}"] \arrow[d, leftrightarrow] & h(D(z_0,r)) \arrow[l, dashed, bend left, "(\text{modulo } i\mathbb{R})"] \\ \text{Analytic functions in } D(z_0,r) & \end{tikzcd}$$



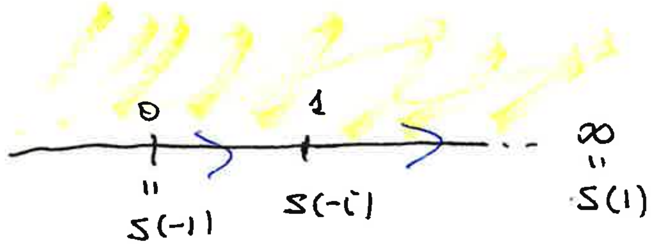
T

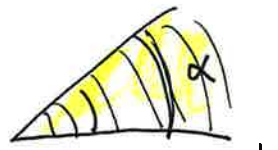
$$\left[-i\,\frac{z-1}{z+1} = T(z) \right]$$



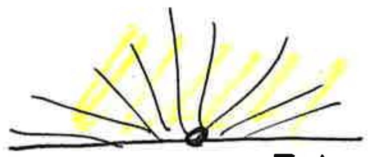
$$S(z) = \frac{z+1}{z-1}$$

$C(z)$





$\longrightarrow$



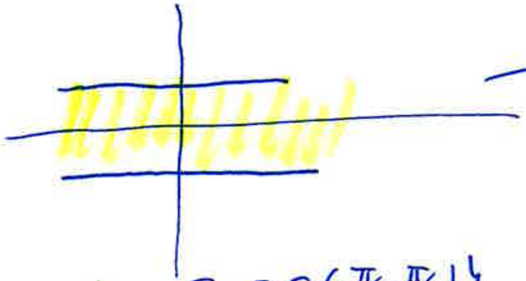
$\xrightarrow{C}$

$z = e^{i\alpha}$

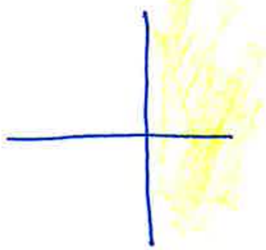
$\alpha \in (0, 2\pi)$

$T(z) = z^{\pi/\alpha} = e^{\frac{\pi}{\alpha} \log_0 z}$

$T'(z) = \frac{\pi}{\alpha} z^{\frac{\pi}{\alpha} - 1}$



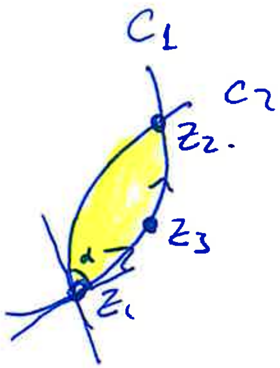
$\xrightarrow{\exp}$



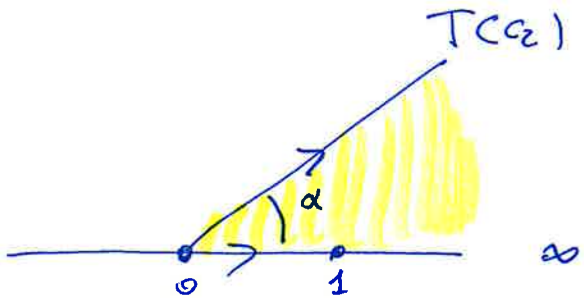
$\xrightarrow{iw}$

$\{z : \operatorname{Im} z \in (-\pi/2, \pi/2)\}$

$w = e^{\operatorname{Re} z} \cdot e^{i \operatorname{Im} z} = e^z$



$\longrightarrow$



$$\left.\begin{array}{l} T(z_1) = 0 \\ T(z_2) = \infty \\ T(z_3) = 1 \end{array}\right\} \quad T(C_1) = \mathbb{R}.$$