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Department of Mathematics
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2. Exercise "Data Structures and Algorithms (DA 4006)"

Problem 1: 7.5p

Provide a pseudocode for converting a non-negative integer into a binary number. You are allowed to use all standard operations and instructions as provided on Slide 8 in the course-slides (*1-Fundamentals-Slides.pdf*). To store the binary number use a (non-dynamic) array where each entry corresponds to one bit and whose size is initialized in terms of the number of bits needed for the binary representation. Explain shortly the basic idea behind your algorithm.

Problem 2: 3+3+3=9p

Let $n, a, b > 1$ be integers. Here, n is treated as variable while a and b are constants. Prove or disprove the following statements. You can use the usual rules applied to log or exponents.

(a) $\log_b(n!) = O(n \log_b n)$.

(b) $\Theta(a^{\log_b(n)}) = \Theta(n^{\log_b(a)})$.

(c) $4^n \in O(2^n)$.

Problem 3: 10p

Given are the following 7 functions:

a) $n \mapsto n\sqrt{n}$

b) $n \mapsto 7n^2 + n\sin(\frac{\pi}{n})$

c) $n \mapsto 2^{\frac{n}{4}}$

d) $n \mapsto (2n+16)(n^2 - \frac{1}{2}n)$

e) $n \mapsto \frac{n}{n+100}$

f) $n \mapsto \sqrt[3]{n} + 2^{\log_2(\frac{3}{n+1})}$

g) $n \mapsto \log_2(n^{2n}) + 1024n$

Arrange the functions in an order $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8$, such that $f_1 \in O(f_2)$, $f_2 \in O(f_3)$, $f_3 \in O(f_4)$, and so on. In addition, provide the asymptotical tightest complexity class $O(g)$ for each function f_i and use here only functions g that are constants or of the form n^k with $k \in \mathbb{Q}_{>0}$, $n\log_2(n)$, or 2^n. *In this exercise, no written proofs need to be provided.*

Problem 4: 2.5+2.5+2.5=7.5p

Given are the following recursions in terms of a function T where $T(1) = \Theta(1)$.

a) $T(n) = 27T(n/3) + 16n^2 + n$

b) $T(n) = 32T(n/2) + n^5 + \log(n)$

c) $T(n) = nT(n-1)$

Determine an exact bound for the behavior of T in each case. In other words, find a function f such that $T \in \Theta(f)$. In cases where the (simplified version of the) Master Theorem is applicable, use it to find f and specify a, b, and d.

Problem 5: 2.5+5+2.5 = 10p

The travelers on a tea trip to Wonderland have all surrendered their passports at the border to obtain a 7-minute visa for Wonderland. The passports, stamped with a Jack of Spades, are now stacked on a pile S, ready to be retrieved by their owners. The White Rabbit stares at the clock anxiously, convinced that everything is taking too long and they will be late for the tea party. The Cheshire Cat suggests that the travelers should queue up in a line Q. The foremost traveler checks if the top passport on the stack belongs to them. If so, they take the passport from the stack, drink the shrinking potion, and leave the queue. Otherwise, the stack remains unchanged, and the traveler returns to the end of the line. This process continues until either the stack or the queue is empty. The program `cheshirecat` looks like this:

```
cheshirecat(stack S, queue Q)
WHILE (Q and S are not empty) DO
    IF (Q.front() = S.top())
        S.pop()
        drinkshrinkdrink(Q.front())
    ELSE
        Q.enqueue(Q.front())
    Q.dequeue()
print("DONE")
```

(a) Which necessary and sufficient condition must S and Q satisfy when calling `cheshirecat` to ensure that the program terminates, i.e., the "`print`("DONE")"command is eventually reached?

(b) Now call `cheshirecat` with $Q = [\text{Theo}, \text{Max}, \text{Paula}, \text{Annemarie}, \text{Otto}]$. S contains the same elements as Q. Provide an initial sequence of elements in S so that `cheshirecat` (i) requires the fewest possible loop iterations and (ii) requires the most possible loop iterations. In both cases, specify the number of iterations.

(c) Consider only instances where the program terminates. Determine $O(\dots)$ for the runtime (asymptotical tight bound) of `cheshirecat` as a function of the problem size n for the worst case. Here, let $n = |S| + |Q|$, the sum of the numbers of elements in S and Q at the time of the call. You can thus assume that $|S|, |Q| \in O(n)$. Justify your answer.

Problem 6: Bonus Exercise 4p

The Curse of the Pharaoh: The ambitious and ruthless archaeologist Dr. Treasurehunter wants to explore the chambers of a newly discovered pyramid. An old inscription states:

In exactly one room lies the Curse of the Pharaoh. Anyone who enters this room will go insane a week later and get green hairs.

Dr. Treasurehunter wants to know as soon as possible which of the n rooms contains the curse. Therefore, he assembles a group of test person. Since Dr. Treasurehunter is stingy and each test person is entitled to financial compensation, he wants this group to be as small as possible. What is the minimum number (depending on n) of test persons needed to obtain the result within a week? Describe precisely which person must enter which rooms and which ones must not. The time required to enter the rooms is negligible.

Bonus problems can be used to earn a certain amount of extra points counted for the final exam.

Deadline: Thursday - April 16, 2026