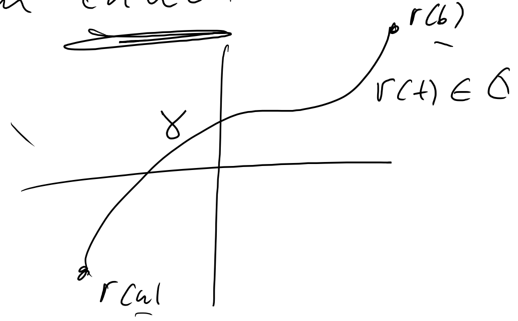
2/5 2022

- Komplexa linjeintegraler
- likformig konvergens

~~Th~~

- definitionen är enkel



$$\int_\gamma f(z)\,dz = \qquad \frac{z = r(t)}{dz = r'(t)\,dt}$$

$$= \int_a^b f(r(t))\cdot r'(t)\,dt$$

Cauchy:

randen
(C^1-styckvis)

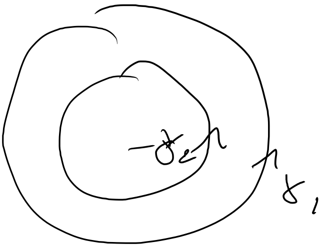
$$\boxed{\int_{\partial D} f(z)\,dz = 0}$$

f(z) är analytisk i en omgivning till D.

CR + Greens sats ger detta.

$$\int_{\partial B} f(z)\,dz = \int_{\gamma_1} + \int_{\gamma_2} = 0$$

$$\Rightarrow \quad \int_{\gamma_1} = -\int_{\gamma_2} = \int_{-\gamma_2}$$



Man kan deformera γ_1 till $-\gamma_2$

$$\int_{\gamma_1} f(z)\,dz = \int_{-\gamma_2} f(z)\,dz$$

* Cauchys integralformel!

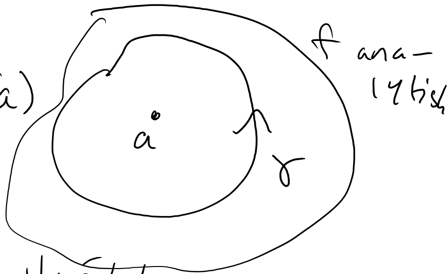
$$\frac{f(z)}{z-a}$$

analytisk funktion

$\lim_{z \to a} \frac{f(z)}{z-a}$ existerar ej.

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz = f(a)$$

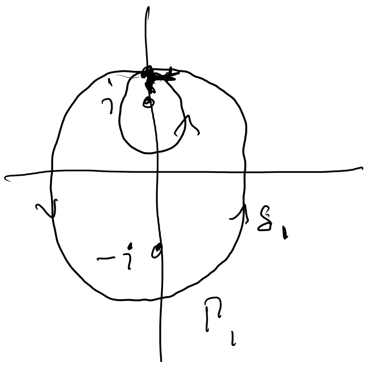
$z=a$ är en polsingularitet.

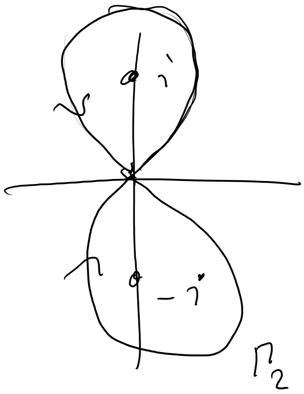


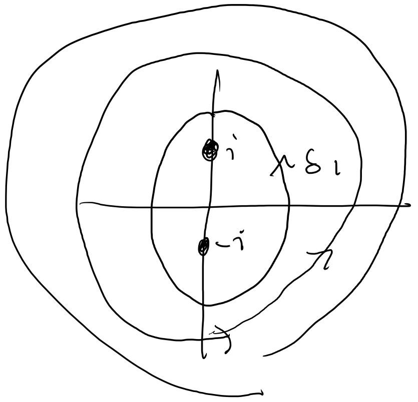
7) $\int_{\Gamma_k} \frac{dz}{1+z^2}$ $\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$

~~$= \frac{A}{z+i}$~~

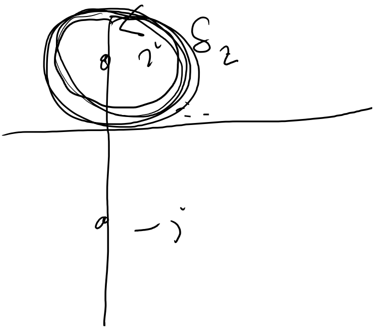
har poler i ±i,







$\int_{\Gamma_1} = \overset{?}{\int_{S_1}} + \boxed{\int_{S_2}}$



pol:

$\frac{1}{z}$ $z = 0$

$\frac{1}{z+i}$ $z = -i$

där funktionen inte är analytisk.

$$\boxed{\frac{1}{2\pi i} \int \frac{\boxed{f(z)}}{z-a} dz = f(a)}$$

$$\frac{1}{2\pi i}\boxed{\int_{\delta_2}\frac{dz}{1+z^2}} = \frac{1}{2\pi i}\int \frac{\frac{1}{z+i}}{z-i}\,dz$$

= analytiskt i en omgivning till $z = i$

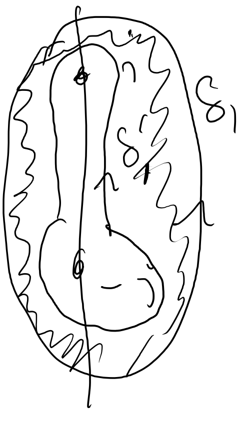
$$= \cancel{\frac{1}{2\pi i}}\,\frac{1}{z+i}\Big|_{z=i} = \cancel{\frac{1}{2\pi i}}\cdot\frac{1}{i+i} =$$

$$= \frac{1}{2i}$$

$$\Rightarrow \int_{\delta_2}\frac{dz}{1+z^2} = \frac{\cancel{2}\pi \cancel{i}}{\cancel{2}\cancel{i}} = \pi$$

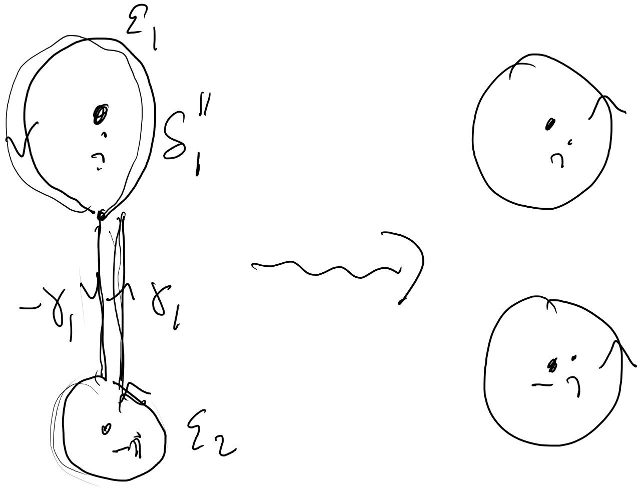
$$\int_{\delta}\frac{f(z)\,dz}{z-a} \qquad \frac{f(z)}{z-a}$$

har residu $\boxed{2\pi i f(a)}$



$$\int_{\delta_1}\frac{dz}{1+z^2} =$$

$$= \int_{\delta_1}\frac{dz}{1+z^2}$$



$$\int_{\delta_1''} = \int_{\varepsilon_1} + \int_{\varepsilon_2} + \cancel{\int_{\gamma_1} + \int_{-\gamma_1}}$$

$$\int_{\delta_1} = \underset{\substack{\shortparallel \\ -i/2\ (\text{tidiquve})}}{\boxed{\int_{\varepsilon_1} \frac{dz}{1+z^2}}} + \int_{\varepsilon_2} \frac{dz}{1+z^2}$$

$$\frac{1}{2\pi i}\int_{\varepsilon_2}\frac{dz}{1+z^2} = \int_{\varepsilon_2}\boxed{\frac{1}{z-i}}\Big|(z+i)\;dz =$$

$$= \cancel{2\pi i}\,\frac{1}{z-i}\Big|_{z=-i}$$

$$\cancel{2\pi i}\,\frac{1}{-i-i} = \frac{\cancel{2\pi i}}{-2i} =$$

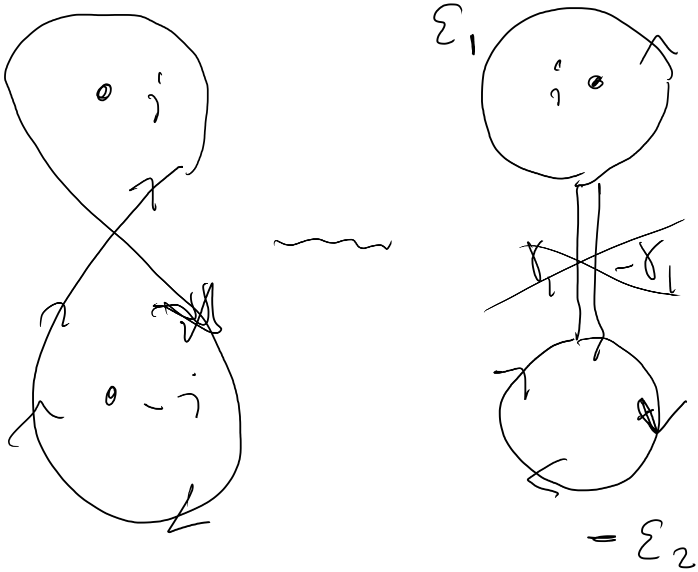
$$= \cancel{-\pi}$$

$$\int_{\varepsilon_2}\frac{dz}{1+z^2} = \frac{2\pi\cancel{i}}{-2\cancel{i}} = -\pi$$

$$\int_{\Gamma_1} = \int_{\delta_2} + \int_{\delta_1} =$$

$$\underset{\pi}{\underset{\shortparallel}{\int_{\delta_2}}} + \underset{\cancel{\pi}}{\underset{\shortparallel}{\int_{\varepsilon_1}}} + \underset{-\cancel{\pi}}{\underset{\shortparallel}{\int_{\varepsilon_2}}}$$

$$= \pi.$$

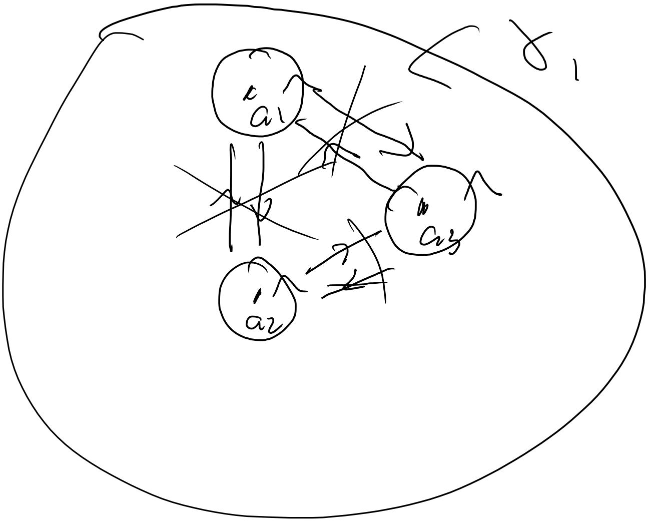


$$\int_{\Pi_2} = \int_{\varepsilon_1} + \int_{-\varepsilon_2} =$$

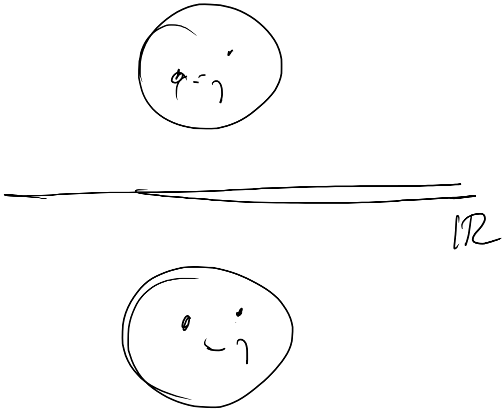
$$= \underset{\overset{\parallel}{\pi}}{\int_{\varepsilon_1}} - \underset{\overset{\parallel}{-\pi}}{\int_{\varepsilon_2}} = \pi - (-\pi)$$

$$= \underline{\underline{2\pi}}$$

$f(z)$ $z = a_1, a_2, a_3$



$$\boxed{\frac{1}{1+z^2}}$$



Hur konvergerar funktionsföljder för att man ska få bra egenskaper hos gränsen?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \dots$$

$$''$$

$$\lim_{n \to \infty} \underbrace{\left(1 + x + \dots + \frac{x^n}{n!}\right)}_{f_n(x)}$$

$f_n(x)$ snäll funktion, kontinuerlig ...

men måste $\lim_{n \to \infty} f_n(x)$ också vara snällt?

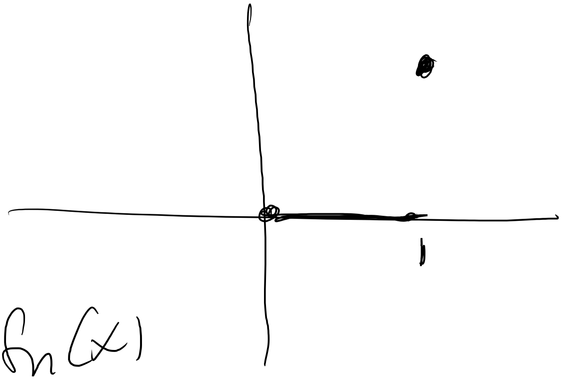
$f_n(x) = x^n$ $\qquad x \in [0,1]$

kont. funkt

$0 \le x < 1$

$\lim_{n\to\infty} f_n(x) = 0$

$\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1 = 1$



$f(x) = \lim_{n\to\infty} f_n(x)$

är en icke-kontinuerlig funktion.

$f_n(x) \longrightarrow f(x)$

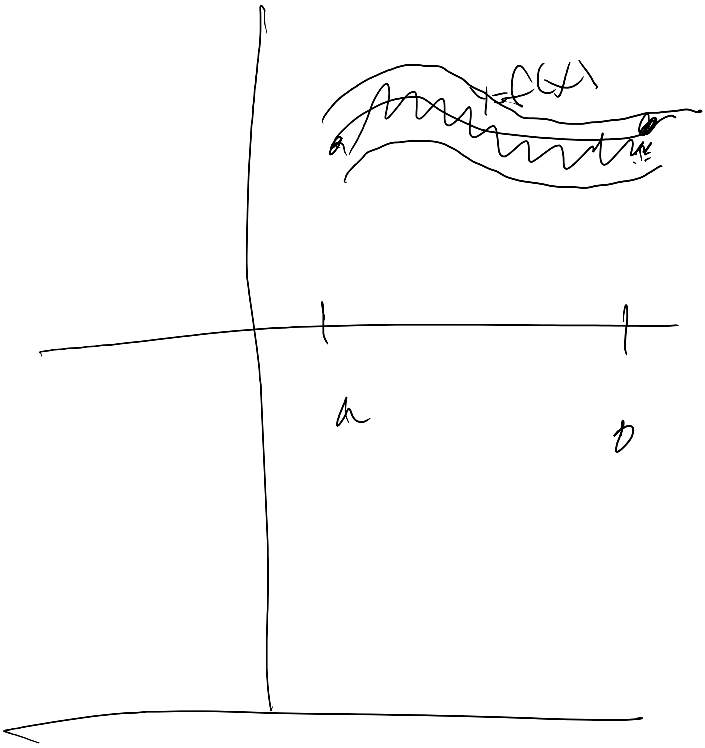
för $\begin{cases} \text{varje } x. \\ \text{tal} \rightsquigarrow \text{tal.} \end{cases}$

likformig konvergens

$f_n(x) \longrightarrow f(x)$

$x \in [a,b]$ som att

$$\sup_{x\in[a,b]} |f(x) - f_n(x)| \to 0$$



för tillräckligt stora n hamnar $f_n(x)$ i en godtycklig liten korridor kring f(x).

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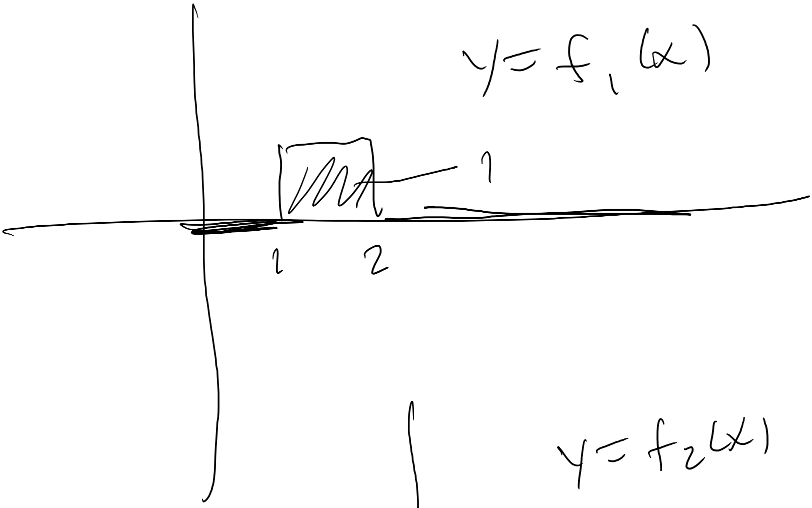
* $f_n(x) \to f(x)$ likformigt

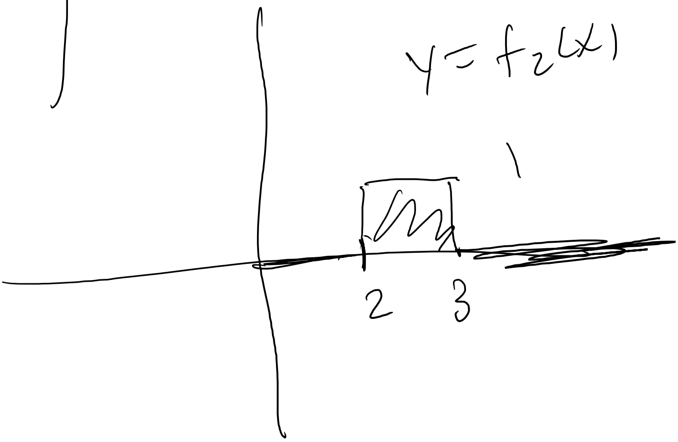
$$\lim_{n\to\infty} \int_a^b \underline{f_n(x)dx} = \boxed{\int_a^b f(x)dx}$$

\- f(x) är kontinuerlig på $[a,b]$.

ex

$$f_n(x) = \begin{cases} 1 & \text{om } x \in [n, n+1] \\ 0 & \text{annars.} \end{cases}$$





$$\lim_{n\to\infty} f_n(x) = f(x) = 0$$

$x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} f_n(x)\,dx = 1$$

$\not\downarrow$ konvergerar inte.

$$\int_{-\infty}^{\infty} f(x)\,dx = 0$$

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = 1 \not\to 0$$

inte likformig konvergens!

ex $f_n(x) = x/n$

$x \in [0,1]$

$$\lim_{n\to\infty} f_n(x) = 0 = f(x)$$

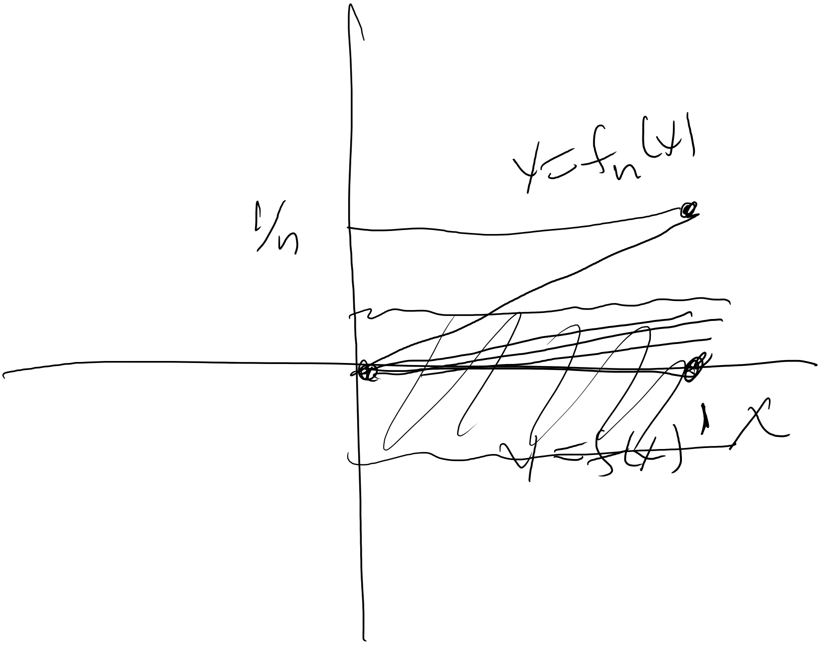
$$\sup_{x\in[0,1]} \Big| \underset{\overset{\shortparallel}{x/n}}{f_n(x)} - \underset{\overset{\shortparallel}{0}}{f(x)} \Big| =$$

$$\sup_{x\in[0,1]} |x/n| = \frac{1}{n}$$

$$\swarrow$$

$$0$$

så vi har likformig konvergens!



likformig konvergens =

lyx, arbetsformen för konvergens