Lecture 14 – Matching

Intro:

Blood type:	A	B	O	AB
Supply	46	34	45	45
Demand	39	38	42	50

A : A, O
B : B O
O : O
AB : any

$\Rightarrow$ Matching theory

Def ~~G~~ bipartite graph. $V = X \cup Y$

a) A matching $M \subseteq E$ no two edge share a common vertex

b) A complete matching ~~of~~ X into Y is a matchin such that $\forall x \in X \ \exists \{x, y\} \in M$.

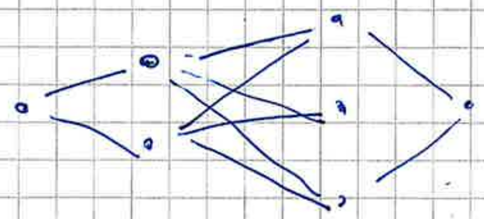
Immediate: $|X| \leq |Y|$

Thm: $G = (X \cup Y, E)$ a complete matching $\exists$
$\Leftrightarrow \forall A \subseteq X \quad |A| \leq |R(A)|$

$$R(A) = \{ y \in Y \mid y \sim x \text{ with } x \in A \}$$

Proof $X = \{x_1 - x_m\}$ $\quad Y = \{y - y_k\}$

We construct a transport network from G by adding a source connected to all vertices in X and a sink connected to all vertices in Y
All edges from source and sink weight 1
All edges ~~between~~ X in E weight $N > |X|$
There is a complete matching $\Leftrightarrow$ There is a a maxflow of capacity $m = |X|$



exit 1 from every $x \in X$

enter 1 from every $y \in Y$

$\Leftarrow$) We will show that $c(P\bar{P}) \geq m$ $\forall$ cut $P\,P^c$ a cut

$A = X \cap P$ $\qquad$ $B = Y \cap P$

$\bar{P} = (A)^c \cup (B^c) \cup \{z\}$

if there is an edge $\{x\,y\}$ with $x \in A$ and $y \in B^c$ then

$$c(P\bar{P}) - w(e) \geq 0$$

$$c(P\bar{P}) \geq w(e) \geq N > |X|$$

Suppose then that no such edge exist.

$c(P\bar{P})$ determined by the weights of edges in from a to A^c & the edges from B to z. weight 1

$c(P\bar{P}) = |X - A| + |B| = m - |A| + |B|$

Observe that

$B \supseteq R(A)$ $\quad$ $y \sim x$ $\quad$ $\{x\,y\}$ is an edge in the cut $y \in B$.

$\Rightarrow c(P\,P^c) \geq m.$

Now $P = \{a\}$ $\rightsquigarrow$ $c(P\,P^c) = m$ so m is the value of the max flow.

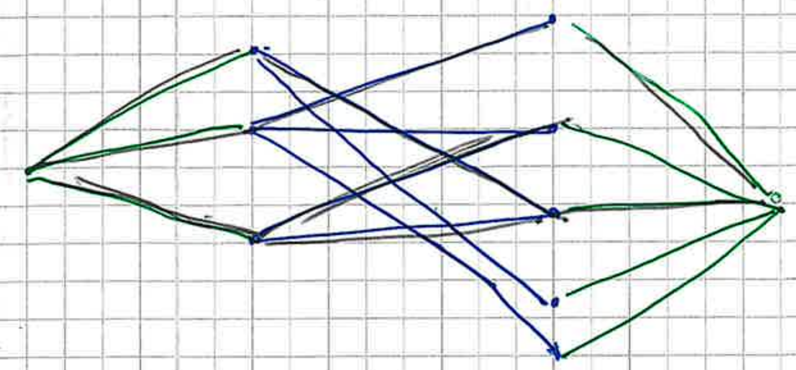
$\Rightarrow$) Suppose otherwise $\exists$ A with $|A| \geq |R(A)|$

$P = A \cup R(A)$ $\qquad$ P^c

$c(P\,P^c) = |X \setminus A| + |R(A)| = |X| - |A| + |R(A)| \leq |X|$

no complete matchi..

Example



$|R(A)|$

$A = \{x\}$ $\quad |R(A)| = \deg(x)$

$A = A' \cup \{x\}$

Cor $G = (X \cup Y, E)$ bipartite.

$\exists$ a complete matching $\Leftarrow$ $\exists\, k \in \mathbb{Z}^+$

$$\deg(x) \geq k \geq \deg(y)$$

$\forall\, x \in X$ and $y \in Y$.

Proof ~~if k does not exist then $\exists\, x$ with~~

($\Rightarrow$) ~~$\deg(y) \leq \deg(x)$ for some x and y.~~

~~$A = \{x\}$ $\quad R(A) = \{y \sim x\}$ $\quad |R(A)| \geq |A|$~~ deg x

($\Leftarrow$) Suppose this is true. the number of ~~vertices~~ edges adjecent to A is $\geq |A|k$

$A \subseteq X$ $\quad |R(A)|$ $\quad$ the number of vertices $\sim A$

is at ~~least~~ most the number of edges $/_k = |A|$

$|R(A)| \geq |A|k/_k = |A|$.

Example 50 students

25 math | each ~~phisics~~ math wanted by

25 Physics | 5 Physics student

every Phisics student has 5 preference as study bu..

Make study pairs

$$\deg m = 5 = \deg p$$

Def: A maximal matching is a matching M such that $\{x \in X \mid \{x,y\} \in M\}$ has maximal cardinality

$G = (V E)$ $\quad V = X \cup Y$ $\quad A \subseteq X$

$\delta(A) = |A| - |R(A)|$ deficiency of A

$\delta(G) := \max\{\delta(A) \mid A \subseteq X\}$ deficiency of G

If $\delta(G) > 0$ then $\nexists$ complete mattching

Thm $G = (X \cup Y, E)$ the maximum number of vertices ~~of~~ in X of a mathcing is
$|X| - \delta(G)$

Proof.
N network. WANT
(a) $c(P\bar{P}) \geq |X| - \delta(G)$
(b) $\exists$ P with $c(PP^c) = |X| - \delta(G)$
$P\ P^c$ a cut
$P = \{a\} \cup A \cup B$ $\quad A = P \cap X \quad B = P \cap Y$

- $(xy) \in N$ $\quad x \in A$ $\quad y \in P \setminus B$ then
$c(P\bar{P}) \geq k \geq |X| \Rightarrow c(P\bar{P}) \geq |X| \geq |X| - \delta(G)$

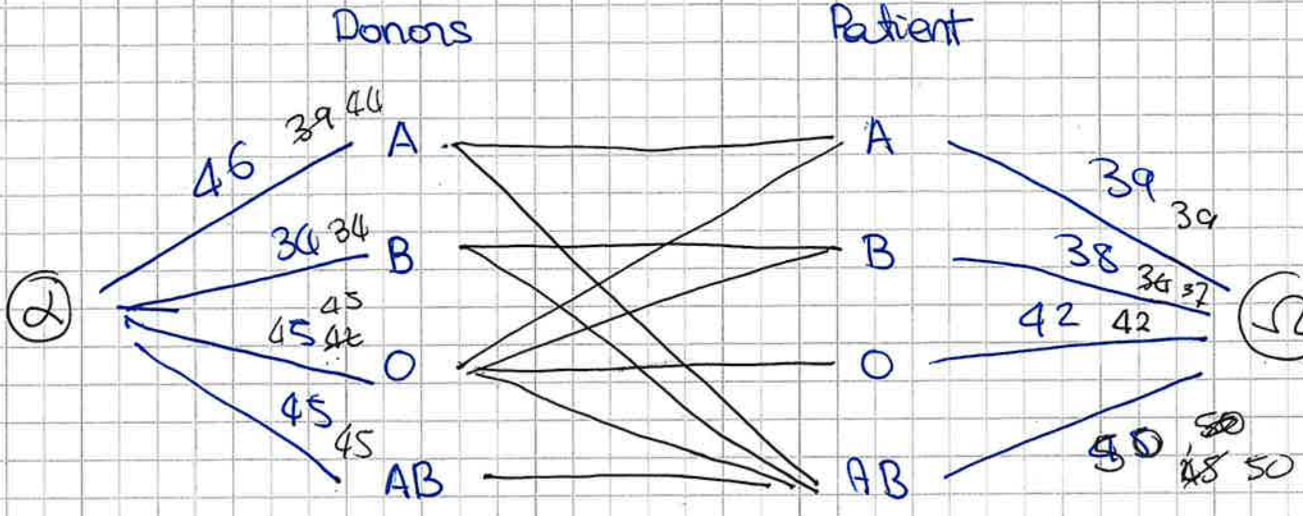
- If no edge as before exist.
$c(PP^c) = |X| - |A| + |B| \geq |X| - |A| + |R(A)|$
$\geq |X| - \delta(A)$
$\geq |X| - \delta(G)$

$\Rightarrow$ (a) ok.

(b) let $A \subseteq X$ with maximal deficiency

$P = A \cup R(A) \cup \{a\}$
$c(P\bar{P}) = |X| - |A| + |R(A)| = |X| - \delta(G)$ #

The initial problem.



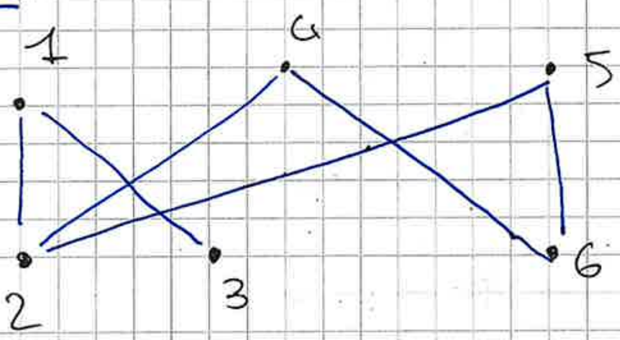
max flow

The value of the max flow is

$$39+37+42+50 < 39+38+42+50$$

$\Rightarrow$ there is no a matching.

<u>Example</u>



note the condition of the corollary fails.

$val(f) = 3 = |\{2, 3, 6\}|$

!!

Proof that it work:

A matching that give as more blod we will have that P(α more flow.

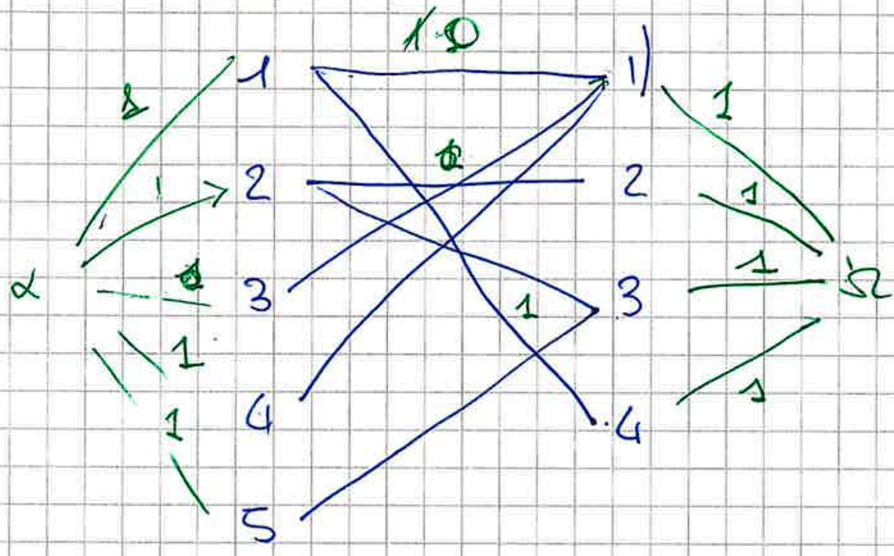
~~Matching in a non bipartite graph.~~

n jobs   n machine

C_{ij} cost of job i if performed by j

Goal: match jobs to machine with lowest co.

Example



- Exist a perfect matching
- Find a maximal matching and $\delta(G$

min cus $\{\alpha\ 3,\ 4\ 1\ \Omega\}$

$$C(P, P^c) = 1 + 3 = 4$$
$$= val(f).$$

Perfect matching $H \subseteq G$ spanning $deg(v) = 1$

- $|V|$ is even. H is bipartite

Thm. $G=(V,E)$ has a perfect matching $\Longleftrightarrow \forall\, U \subseteq V$ the subgraph induced by $V \setminus U$ has at most $|U|$ connected cp with an odd number of vertices

Proof $S \subseteq V$ $G-S$ = graph induced by $V \setminus S$
$o(G)$ = # conn cp of $G-S$ with odd number of vertic

$\Rightarrow$) $U \subseteq V$ C an odd cp of G which has a perfect matching: $\exists\, y \in C$ $y \sim_M u \in U$

$|o(G-U)| \leq |U|$ as one vertex cannot be matched to more than 1 cp

$\Leftarrow$) bad set: $S \subseteq V$ violating the condition
G w/out match $\Rightarrow$ find a bad set
- $|G| =$ ~~even~~ odd then $\emptyset \subseteq V$ bad.
- $|G|$ edge maximal (any edge would create a PM)

$G' = G + e$ S edge ~~maximal for~~ bad for G'

then $S \subseteq V$ is bad for G:

any odd component of $G-S$ is a union of components of $G'-S$

~~→ e is in an even component of G'−S~~

an odd cp is union of odd cp of $G-S$

one is odd $|S| \leq o(G'-S) \leq o(G-S)$.

Claim: S is bad $\Leftarrow$) $G-S$ union of disjoint cliques every $s \in S$ has deg $n-1$

$\Rightarrow$) S bad $G-S$ not complete. adding an edge to the component makes a perfect match..

but adding an edge in a odd cp does not change the parity

S has a vertex of smaller degree.

we add a missing edge, it does not ~~help~~. chan odd even compone.

but contradict maximality

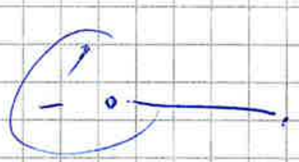
$\Leftarrow$) Assume S not bad. If $S = \emptyset$ then

$o(G) = 0$. ~~we can assume~~ $G-S$ is a disjoint union of complete graph of even order $\Rightarrow \exists$ perfect matching.

If $S \neq \emptyset$

$\forall$ O $\in G-S$ odd cp $\exists a \in S$.

pair all the ~~even~~ other. pair up all the remaining vertices in S. $|V|$ even



$o(S)$ even

$\Rightarrow$ even number of O'

$o(S)$ odd . .

We assume G no perfect match.

$S = \{ v \in G \mid v \sim w \ \forall w \in G \}$ $\deg(v) = |V| - 1$

If $\cancel{G} = G - S$ is not a union of complete graps then

$G-S$ has $x\,y$ ~~adjacent~~ non adjacent.

$x\,a\,b$ first 3 vertices in the shortest path $x - y$

$(x\,b) \notin E$ $a \notin S$ then $\exists c$ such that $(a\,c) \notin E$

(shortest $\exists M_1$ on $G + (x\,b)$ M_2 on $G + (a\,c)$

P edges of maximal path in $\cancel{G}$ from c wit an edge in M_1 and who alternate edge. En is we arrive at

$x\,a\,b$ adjacent not in G

v last vertex e last edge

$\cancel{e} \in M_1 \Rightarrow v = a$ P(ac) cycle

$M_2 \Rightarrow v = \{x\,b\}$ P(va)(ac)

even cycle in $G + (a\,c)$ with every ot edge in M_2 all vertex matched by M_2 or M_1