Grothendieck AB1–5 (6) $\mathcal{A}$ additive

(Pre) AB1 : $\mathcal{A}$ has kernels & Cokernels

(Pre) AB2 : $\mathcal{A}$ AB1 + the isomorphism on images & coimages

AB3 : AB2 + ∞ products (cocomplete)

AB3* : AB2 + ∞ coproducts / direct sums (complete)

Ex : R Modules

$\neg$ Ex : fg modules / finite abelian groups

AB4 : AB3 + direct sums are exact

R-mod, $\mathcal{O}_X$ mod

$\varphi_i : M_i \xrightarrow{\text{mon}} N_i$ $\quad \oplus M_i \quad \oplus N_i$ exists

$\Rightarrow \oplus \varphi_i : \oplus M_i \longrightarrow \oplus N_i$ MONO

AB4* : AB3* + ~~pa~~ direct product are exact

$(R\text{ mod})^{op}$ $\quad \neg$ Ex Sheaves of Ab on a general top space

FOR AB5 we need limits & colimits

(McLane) Cathegory theory in context.

$\mathcal{J}$ $\mathcal{C}$ two categories

$\mathcal{C}^{\mathcal{J}} = \{ F : \mathcal{J} \to \mathcal{C} \text{ functors} \}$

$\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{J}}$

$c \longmapsto \{ F(j) = c \quad F(\varphi : i \to j) = 1_c \}$

$f : c \to c'$

$$\begin{tikzcd} i \arrow[d] & \Delta c(i) = c \arrow[r, "f"] \arrow[d, "1_c"'] & \Delta c'(i) = c' \arrow[d, "1_{c'}"] \\ j & \Delta c(j) = c \arrow[r, "f"] & \Delta c'(j) = c' \end{tikzcd}$$

F functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ a colimit ~~of~~ for F

(direct limit) is $(\varinjlim F, u)$ $\quad \varinjlim F \in \mathrm{Ob}\,\mathcal{C}$

$u : F \longrightarrow \Delta(\varinjlim F)$

$$\begin{tikzcd} F \arrow[r, "u"] \arrow[dr, "\forall"'] & \Delta(\varinjlim F) \arrow[d, dashed, "\exists !"] \\ & \Delta C \end{tikzcd}$$

$j \in \mathcal{J}$

$$\begin{tikzcd} j \arrow[d, "f"] & u_j : F(j) \arrow[r] \arrow[d, "F(f)"] & \varinjlim F \arrow[d, "\mathbb{1}_{\varinjlim F}"] \\ i & u_i : F(i) \arrow[r] \arrow[ur] & \varinjlim F \end{tikzcd} \quad \text{in } \mathcal{C}$$

$\forall \quad \{\tau_i\}$

$$\begin{tikzcd} F(j) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & c \\ F(j) \arrow[ur] & \end{tikzcd}$$

$\exists! \quad \varphi : \Delta c \longrightarrow \Delta \varinjlim F$

$\varphi : c \longrightarrow \varinjlim F$

Limit (inverse limit)

$(\varprojlim F, \ v : \Delta \varprojlim F \longrightarrow F)$

$$\begin{tikzcd} \Delta \varprojlim F \arrow[rr, "v"] & & F \\ & \Delta c \arrow[ul, "\exists !"] \arrow[ur, "\forall"'] & \end{tikzcd}$$

A filtered category $\mathcal{J}$ is a category (usually small)

$\mathcal{J} \neq \emptyset \qquad \forall i, j \in \mathrm{Ob}\,\mathcal{J}$

$$\exists k \in \mathcal{J} \qquad \begin{tikzcd} & i \\ k \arrow[ur] \arrow[dr] & \\ & j \end{tikzcd}$$

$\mathcal{J}$ filtered $\Rightarrow$ filtered limits & colimit

<u>AB5</u> : AB3 + filtered colimit are exact

<u>AB5*</u> : AB3 + filtered limits are exact

$\neg$ Ex ~~R~~- mod R   R not noetherian

A category $\mathcal{C}$ has a generator U if for every

$N \hookrightarrow M$

$\exists\, U \to M$ that does not factor through $N \hookrightarrow M$

AB5 + $\exists$ generator = "GROTHENDIECK" category.

CATEGORY OF COMPLEXES

$\mathcal{A}$ abelian (not nec small ☹)

a complex (cochain)

$$\begin{tikzcd}
X^\bullet : \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dr, phantom, "\circlearrowright"] & X^n \arrow[r, "d_X^n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowright"] & X^{n+1} \arrow[r, "d_X^{n+1}"] \arrow[d, "f_{n+1}"] & \cdots \\
Y^\bullet : \arrow[r] & Y^{n-1} \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

$d_X^{n+1} \cdot d_X^n = 0$

$$\mathrm{Hom}_{\mathcal{A}}(X^\bullet, Y^\bullet) = \{ f_i : X^i \to Y^i \ \circlearrowright \}$$

$\mathrm{Kom}(\mathcal{A})$ — Ob Kom $\mathcal{A}$ complexes of objects of $\mathcal{A}$

Hom ~~(Kom~~ Kom $\mathcal{A}$ are the morphisms of complexes

$\hookrightarrow$ Abelian. $\varphi : X^\bullet \longrightarrow Y^\bullet$

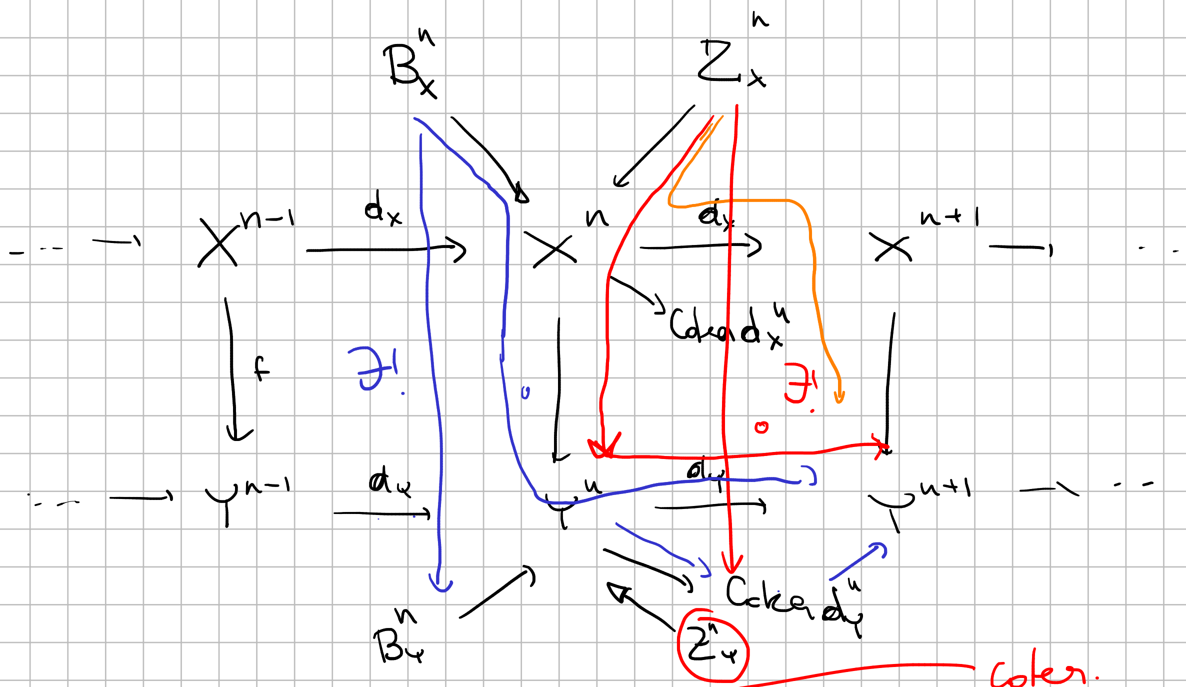
$(\mathrm{Ker}\, \varphi_i, k_i) \rightsquigarrow$ kernel φ

<u>Cohomology</u>

$$\begin{tikzcd}
B_X^n \ \mathrm{Ker}\,\mathrm{Coker}\, d_X \arrow[rr] \arrow[dr] \arrow[ddr, red] & & \mathrm{Ker}\, d_X^n \ Z_X^n \arrow[dl] \\
X^{n-1} \arrow[r] & X^n \arrow[r] \arrow[d] & X^{n+1} \\
& \mathrm{Coker}\, d_X^{n-1} &
\end{tikzcd}$$

$H^n(X^\bullet) = \mathrm{Coker}(\longrightarrow)$

If $f : X^\bullet \longrightarrow Y^\bullet$ is a morphism then there is a map induced in cohomology



coker.

$$\begin{tikzcd}
B_X^n \arrow[r] \arrow[d] & Z_X^n \arrow[r] \arrow[d] & H^n(X) \arrow[r] \arrow[d, "\exists!"] & 0 \\
B_Y^n \arrow[r] & Z_Y^n \arrow[r] & H^n(Y) \arrow[r] & 0
\end{tikzcd}$$

<u>Homotopy</u>: $f: X^\bullet \to Y^\bullet$ complexes in $\mathcal{A}$

$f \in \mathrm{Hom}_{\mathrm{Kom}(A)}(X^\bullet, Y^\bullet)$

We say that $f \sim 0$ (homotopic to 0)

if $\exists\, k^i : X^i \longrightarrow Y^{i-1}$

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-2} \arrow[r] \arrow[d] & X^{n-1} \arrow[r] \arrow[d] \arrow[dl] & X^n \arrow[r] \arrow[d, "f^n"] \arrow[dl] & X^{n+1} \arrow[r] \arrow[d] \arrow[dl] & \cdots \\
\cdots \arrow[r] & Y^{n-2} \arrow[r] & Y^{n-1} \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

$$f^n = d_Y^{n-1} k^n + k^{n+1} d_X^n$$

$f \sim 0 \qquad m \in \mathbb{Z} \qquad m \cdot f \sim 0$

$f, g \sim 0 \implies f+g \sim 0$

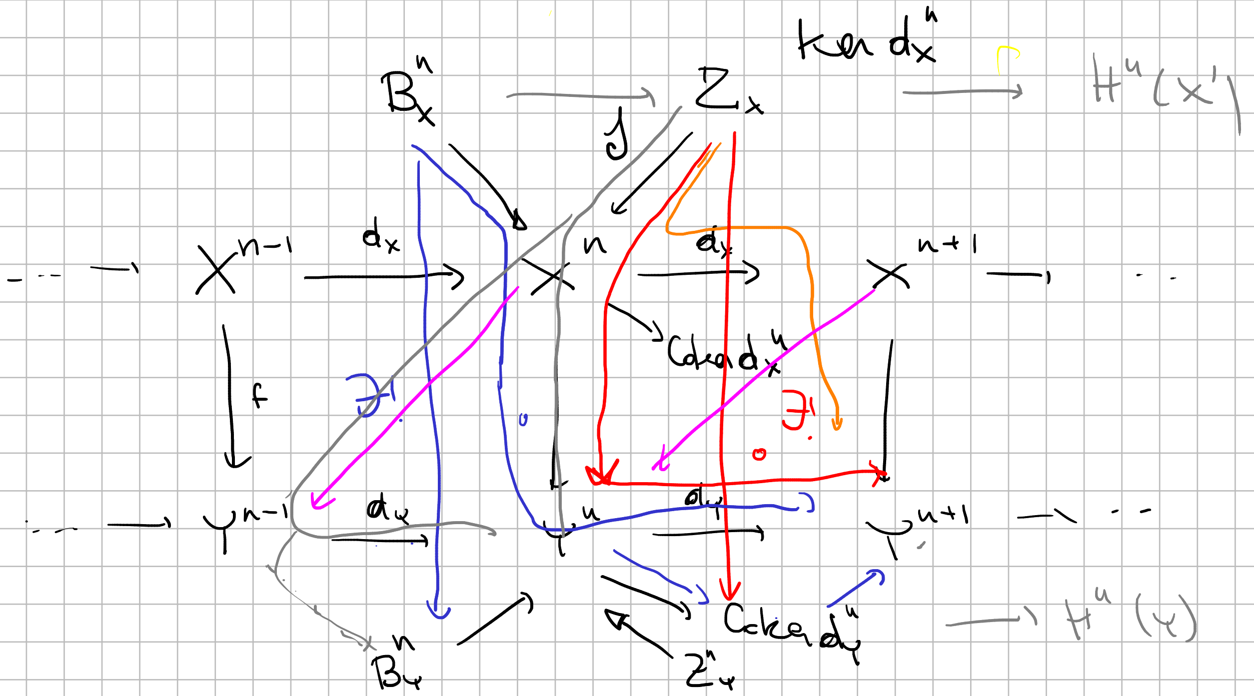
$f \sim g \overset{\text{def}}{\iff} f-g \sim 0$

New category $K(A)$ $\qquad \operatorname{Ob} K(A) = \operatorname{Kom} A$

$\operatorname{Hom}_{K(A)}(X^\bullet, Y^\bullet) = \operatorname{Hom}_{\operatorname{Kom}(A)}(X^\bullet, Y^\bullet) / \sim 0$

NOT ABELIAN
(in general)

Lem: $f \sim 0$ then the map induced in cohomology is 0



$$f^i = k^{i+1} d_X + d_Y k^i$$

$$f \cdot j = \underbrace{k^{i+1} d_X j}_{0} + d_Y k^i j$$

$$d_Y f j = d_Y^2 k^i j$$

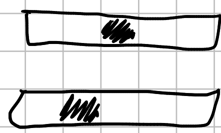
Shift functors $\quad [k] : \operatorname{Kom} A \longrightarrow \operatorname{Kom} A$

$$(X^\bullet[k])^n = X^{n+k}$$

$$(f[k])^n = f^{n+k}$$

Ex $X^\bullet$

$$\xrightarrow{0} X \xrightarrow{0}$$
$\uparrow$ deg 0



$$(X^\bullet[k])^n = X^{n+k} = 0 \quad \text{if } n \neq -k$$
$$= X \quad n = -k$$

$X^\bullet$: $\xrightarrow{0} X \longrightarrow 0$

$\uparrow$ deg $-k$

$$H^k(X^\bullet) = H^0(X[k])$$

Prop Given $0 \to X_1^\bullet \to X_2^\bullet \to X_3^\bullet \to 0$ short exact sequence.
then we have a long exact sequence
$$\longrightarrow H^n(X_1^\bullet) \longrightarrow H^n(X_2^\bullet) \longrightarrow H^n(X_3^\bullet) \longrightarrow H^{n+1}(X_1) \longrightarrow \cdots$$

Ingredients

Snake lemma:

$$\begin{tikzcd}
 & K_X \arrow[r] & K_Y \arrow[r] & K_Z & \\
 & X \arrow[r] \arrow[d, "\alpha"'] & Y \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] & \\
 & C_1 \arrow[r] & C_2 \arrow[r] & C_3 &
\end{tikzcd}$$

exact sequence

$$\mathrm{Ker}\,\alpha \to \mathrm{Ker}\,\beta \to \mathrm{Ker}\,\gamma \longrightarrow \mathrm{Coker}\,\alpha \to \mathrm{Coker}\,\beta \to \mathrm{Coker}\,\gamma$$

Apply

$$\begin{tikzcd}
 & \mathrm{Coker}\, d_{X_1}^{n-2} \arrow[r] \arrow[d] & \mathrm{Coker}\, d_{X_2}^{n-2} \arrow[r] \arrow[d] & \mathrm{Coker}\, d_{X_3}^{n-2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}\, d_{X_1}^n \arrow[r] & \mathrm{Ker}\, d_{X_2}^n \arrow[r] & \mathrm{Ker}\, d_{X_3}^n &
\end{tikzcd}$$

We $\quad \|$

$$H^{n-2}(X) \to \mathrm{Coker}\, d^{n-2} \longrightarrow \mathrm{Ker}\, d^n \longrightarrow H^n(X)$$

$X \in A$ $\qquad$ $y \in X$

equivalence classes

(Y, h)

$0 \in X$
$\shortparallel$
$(0, 0 : \cancel{0} \to X)$

$h : Y \to X$

Z $\qquad$ Y' $\quad$ h'

$f \qquad f(y) = (Y\ f \cdot h)$

$$\begin{array}{c} f : X \longrightarrow T \\ h \uparrow \\ Y \end{array}$$

Rules

1) f mono $\iff$ $f(y) = 0 \implies y = 0$
2) f mon $\iff$ $f(y_1) = f(y_2) \implies y_1 = y_2$
3) f epi $\iff$ surjective on element
4) $f : X \to T$ is 0 $\iff$ $f(y) = 0 \quad \forall y \in X$

~~5) $f(x) = f(x')$ $\iff$ $\exists$ $z \in X$ $f(z) = 0$~~
~~$\shortparallel$ "$x - x'$"~~
~~$f : X \to T$~~
~~for all $g : T \to S$~~

$\mathrm{Kom}^{*}(A) \qquad * \in \{+, -, b\}$

$\mathrm{Ob}\,\mathrm{Kom}^{+}(A) = \{X^n = 0 \quad \text{when } n \ll 0\}$

$-$ $\qquad\qquad$ $n \gg 0$

b