Derived categories Lecture 4

1) "Resolutions" are unique in the $K(\mathcal{A})$
2) The Homotopy category is triangulated

Recall $\mathcal{A}$ abelian $\text{ob}\, K(\mathcal{A}) : \text{ob}\, \text{Kom}(\mathcal{A})$

$$\text{Hom}_{K(\mathcal{A})}(A,B) = \text{Hom}_{\text{Kom}(\mathcal{A})}(A,B)_{\sim}$$

Projective Resolutions $\mathcal{A}$ has enough projectives

$X \in \mathcal{A}$ $\exists$ P^0 proj $P^0 \xrightarrow{\varepsilon_X} X$

$(K\ k)$ Kernel of ε_X

$$P^{-1} \twoheadrightarrow K$$

$$P^{-1} \xrightarrow{\delta^{-1}} P^0 \xrightarrow{\varepsilon_X} X$$

Reiterate $\boxed{\cdots P^{-n} \to P^{-n+1} \to \cdots \to P^{-1} \to P^0} \twoheadrightarrow X$

$P^0 \to 0$

$P^\bullet$

$H^i(P^\bullet) = 0$ for $i \neq 0$

$H^0(P^\bullet) = X$

$P^\bullet$ is a projective resolution of X

a complex of projective objects $P^\bullet$ with a map

$P^\bullet \xrightarrow[\varepsilon_X]{} X$ such that induces

isomorphisms in cohomology

Thm (Gelfand-Manin)

$f : X \to Y$ morphism in $\mathcal{A}$

$P^\bullet \xrightarrow[\varepsilon_X]{} X$ $Q^\bullet \xrightarrow[\varepsilon_Y]{} Y$ projective res.

$\exists$ a morphism $R(f) : P^\bullet \to Q^\bullet$ such that

$$\begin{tikzcd} P^0 \arrow[r, "\varepsilon_X"] \arrow[d, "R(f)^0"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d] \\ Q^0 \arrow[r, "\varepsilon_Y"'] & Y \end{tikzcd}$$

$R(f)$ is unique up to Homotopy equiv.

Proof :

$$\begin{tikzcd}
P^0 \arrow[r, "\varepsilon_X"] \arrow[d, "R(f)^0"'] \arrow[dr] & X \arrow[d, "f"] \\
Q^0 \arrow[r, two heads] & Y
\end{tikzcd}$$

P^0 projective

Inductive step : $\mathrm{Hom}_A(P^{-i}, -)$ is exact if P^{-i} proj.

$$\begin{tikzcd}
Q^{-i} \arrow[r] & Q^{-i+1} \arrow[r] & Q^{-i+2} \arrow[r] & \cdots & Q^0 \arrow[r, "\varepsilon_Y"] & Y \\
P^{-i} \arrow[r] \arrow[u, "R(f)^{-i}"] \arrow[ur, "\gamma"] & P^{-i+1} \arrow[r] \arrow[u] \arrow[ul, phantom, "\circlearrowleft"] & P^{-i+2} \arrow[r] \arrow[u] \arrow[ul, phantom, "\circlearrowleft"] & \cdots & P^0 \arrow[r, "\varepsilon_X"] & X
\end{tikzcd}$$

$$\begin{array}{ccccc}
 & & \gamma & \longmapsto & 0 \\
\mathrm{Hom}(P^{-i}, Q^{-i}) & \xrightarrow{\ \circ d_Q^{-i}\ } & \mathrm{Hom}(P^{-i}, Q^{-i+1}) & \xrightarrow{\ d_Q^{-i+1}\ } & \mathrm{Hom}(P^{-i}, Q^{-i+2}) \quad \text{exact} \\
 & & & \longmapsto & 0
\end{array}$$

$\exists\ R(f)^{-i} \in \mathrm{Hom}(P^{-i}, Q^{-i})$ such that

$$d_Q^{-i} R(f)^{-i} = \gamma - R(f)^{-i+1} d_P^{-i}$$

$$\begin{tikzcd}
Q^{-i} \arrow[r] & Q^{-i+1} \arrow[r] & Q^{-i+2} \arrow[r] & \cdots & Q^0 \arrow[r, "\varepsilon_Y"] & Y \\
P^{-i} \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & P^{-i+1} \arrow[r] \arrow[u, shift left] \arrow[u, shift right] \arrow[ul, dotted] & P^{-i+2} \arrow[r] \arrow[u, shift left] \arrow[u, shift right] \arrow[ul, dotted] & \cdots & P^0 \arrow[r, "\varepsilon_X"] & X
\end{tikzcd}$$

$\alpha = R(f)^0 - (R(f)')^0$

epi + proj $\Rightarrow \exists\ k^0$

$$\begin{tikzcd}
Q^{-1} \arrow[r] & Q^0 \arrow[r, two heads] & 0 \\
P^{-1} \arrow[r] & P^0 \arrow[r] \arrow[u, "\alpha"] \arrow[ul, "k^0"] & 0 \arrow[u, "\varphi"] \arrow[ul]
\end{tikzcd}$$

$$d_Q^{-1} k^0 + \cancel{0\, d_\varphi} = \alpha$$

$$
\begin{array}{ccccccccccc}
Q^{-i-1} & \longrightarrow & Q^{-i} & \longrightarrow & Q^{-i+1} & \longrightarrow & Q^{-i+2} & \longrightarrow \cdots & Q^{0} & \xrightarrow{\varepsilon_Y} & Y \\
 & & \uparrow & \nwarrow & \uparrow \alpha^{-i+1} & \nwarrow & \uparrow & & & & \\
P^{-i-1} & \longrightarrow & P^{-i} & \longrightarrow & P^{-i+1} & \longrightarrow & P^{-i+2} & \longrightarrow \cdots & P^{0} & \xrightarrow{\varepsilon_X} & X
\end{array}
$$

$$\alpha^{-i} = R(f)^{-i} - \left(R(f)'\right)^{-i}$$

$$d_Q(\alpha^{-i} - k^{-i+1} d_P) = 0$$

$$d_Q(\alpha^{-i}) = \alpha^{-i+1} d_P = (d_Q k^{-i+1} + k^{-i} \circ d_P) d_P$$

$$= d_Q(k^{-i+1} \circ d_P)$$

$$\gamma \longmapsto 0$$

$$\mathrm{Hom}(P^{-i}, Q^{-i}) \longrightarrow \mathrm{Hom}(P^{-i}, Q^{-i+1})$$

I can construct k^{-i} as befor.

<u>CoR</u> $f = id : X \longrightarrow X$

Every "bech" resolution is homotopic equivalent to a projective one.

Given $P^{\bullet}$ $Q^{\bullet}$ proj resolution of X

$$P^{\bullet} \xrightarrow{\varepsilon_X} X \leftarrow Q^{\bullet} \qquad \exists\ R(id) : P^{\bullet} \longrightarrow Q^{\bullet}$$

The map induced in cohomology $= id$

$$X \in A \rightsquigarrow 0 \to 0 \to X \to 0 \to 0 \in K(A)$$

$$
\begin{array}{ccccccc}
P^{-1} & \longrightarrow & P^{0} & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow 0 & & \downarrow \varepsilon_X & & \Leftarrow & & \text{morphism in } K(A) \\
0 & \xrightarrow{0} & X & \longrightarrow & 0 & &
\end{array}
$$

Induces an isomorphism in cohomology.

Def $f: X^\bullet \longrightarrow Y^\bullet$ is a quasi-isomorphism if induces an isomorphism in cohomolog (at every level.

Mapping Cone (Kashiwara - Shapira sheaves on Manifold)

$f: X^\bullet \longrightarrow Y^\bullet$ in $\mathrm{Kom}(\mathcal{A})$

The cone of f

$\mathrm{Kom}(\mathcal{A}) \ni C(f)$

$$\begin{array}{c} C(f)^n = X^{n+1} \oplus Y^n \\ \Big\downarrow d_{C(f)} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} \\ C(f)^{n+1} = X^{n+2} \oplus Y^{n+1} \end{array}$$

$\alpha(f): Y^\bullet \longrightarrow C(f)$

$\alpha^n = \begin{pmatrix} 0 \\ id_{Y^n} \end{pmatrix}$

$\beta(f): C(f) \longrightarrow X^\bullet[1]$

$\beta(f)^n = (id_{X^{n+1}}\ 0)$

Lemma: For any morphisms of complexes $f: X^\bullet \to Y^\bullet$
Then is $\phi: X[1] \longrightarrow C(\alpha(f))$ inducing an homotopy equivalence (iso in $K(\mathcal{A})$)
In addition we have the following commutative diagram in $K(\mathcal{A})$

$$\begin{tikzcd} Y \arrow[r, "\alpha(f)"] \arrow[d, "id"'] & C(f) \arrow[r, "\beta(f)"] \arrow[d, "id"] & X[1] \arrow[r, "-f[1]"] \arrow[d, "\phi"] & Y[1] \arrow[d, "id"] \\ Y \arrow[r, "\alpha(f)"] & C(f) \arrow[r, "\alpha(\alpha(f))"'] & C(\alpha(f)) \arrow[r] & Y[1] \end{tikzcd}$$

Proof :

$$\begin{tikzcd}
X[1] & X^{n-1} \arrow[r, "-d_X"] & X^{n} \arrow[r, "-d_X"] & X^{n+1} \arrow[d] \arrow[d, bend left, "-f"] \\
 & & & Y^n \oplus X^{n+1} \oplus Y^{n+1}
\end{tikzcd}$$

$$C(\alpha(f))^n = C(f)^n \oplus Y^{n+1} = Y^n \oplus X^{n+1} \oplus Y^{n+1}$$

$$C(f) \oplus Y[1]$$

$$\begin{pmatrix} 0 \\ id \\ -f^{n+1} \end{pmatrix} = \phi^n$$

this is a map of complexes

$$\begin{tikzcd}
X^{n} \arrow[r] \arrow[d] & X^{n+1} \arrow[d] \\
C(\alpha(f))^n \arrow[r] & C(\alpha(f))^{n+1}
\end{tikzcd}$$

$$\psi^n : C(\alpha(f))^n \longrightarrow X[1]^n$$

$$Y^n \oplus X^{n+1} \oplus Y^{n+1} \qquad\qquad X^{n+1} \qquad\qquad (0 \quad id_{X^{n+1}} \quad 0)$$

- $\psi \circ \phi = id_{X[1]}$

$$\psi \circ \alpha(\alpha(f)) = \beta(f)$$

$$\beta(\alpha(f)) \circ \phi = -f[1]$$

$$\phi \circ \psi \sim id \qquad Y^n \oplus X^{n+1} \oplus Y^{n+1}$$

$$\begin{tikzcd}
C(\alpha(f))^{n-1} \arrow[r] \arrow[d, "\psi^{n-1}"] & C(\alpha(f))^{n} \arrow[r] \arrow[d, "\psi^{n}"] \arrow[dd, bend left, "id"] & C(\alpha(f))^{n+1} \arrow[r] \arrow[d, "\psi^{n}"] & {} \\
X^{n} \arrow[r] \arrow[d, "\phi"] & X^{n+1} \arrow[r] \arrow[d, "\phi^{n}"] & X^{n+2} \arrow[d, "\phi^{n}"] & \\
C(\alpha(f))^{n-1} \arrow[r] & C(\alpha(f))^{n} \arrow[r] & C(\alpha(f))^{n+1} &
\end{tikzcd}$$

$$Y^{n-1} \oplus X^{n} \oplus Y^{n}$$

A triangle (Δ) in $K(A)$ is a sequence of objects & morphis

$$\begin{tikzcd} & Z \arrow[dl, "+1"'] & \\ X \arrow[rr] & & Y \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[r] \arrow[d, "h"] \arrow[dr, phantom, "\circlearrowright"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

A triangle is distinguished if it is isomorphic to

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} C(f) \xrightarrow{\beta(f)} X[1]$$

PROPERTIES:

TR0 Any triangles isom to a distinguished one is distinguished

TR2 Any morphism $f: X \to Y$ can be completed to a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

$$Z = C(f)$$

TR3 Rotation

$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ distinguished

then

$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{f[1]} Y[1]$ distinguished

this is the lemma

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r, "h"] \arrow[d] & X[1] \arrow[d] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "\alpha(f)"] & C(f) \arrow[r, "\beta(f)"] & X[1] \end{tikzcd}$$

Lemma $\Rightarrow$

$Y' \longrightarrow C(f') \longrightarrow X'[1] \longrightarrow Y'[1]$ distinguished

TR1

$X \xrightarrow{id} X \longrightarrow 0 \longrightarrow X[1]$ is distinguished

the cone $C(0 \longrightarrow X) \cong X$

$$0 \oplus X^n$$

$$\begin{pmatrix} 0 & 0 \\ 0 & d_{X^n} \end{pmatrix}$$

$0 \longrightarrow X \xrightarrow{id} X \longrightarrow 0[1]$ is distinguished

ROTATE

$X \longrightarrow X \longrightarrow 0 \longrightarrow X[1]$ distinguished

TR4 (5 lemma)

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, dashed, "\exists"] & X[1] \arrow[d, "u[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

dist

$$Z = C(f) \overset{n}{\rightsquigarrow} X^{n+1} \oplus Y^n$$

$$\downarrow \quad \begin{pmatrix} u^n & 0 \\ s & v^n \end{pmatrix}$$

$$Z' \overset{n}{\rightsquigarrow} (X')^{n+1} \oplus (Y')^n$$

Given by the homotopy that makes the diagram commut

TR5 (oh No!!!)

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id"'] & Y \arrow[r] \arrow[d, "g"] & C(f) \arrow[r] \arrow[d] & X[1] \arrow[d, "id"] \\
X \arrow[r, "g \circ f"] \arrow[d, "f"'] & Z \arrow[r] \arrow[d, "id"] & C(g \circ f) \arrow[r] \arrow[d] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"] \arrow[d, "\alpha(f)"'] & Z \arrow[r] \arrow[d, "\alpha(g \circ f)"] & C(g) \arrow[r] \arrow[d, "\alpha(g)"] & Y[1] \arrow[d, "\alpha(f)[1]"] \\
C(f) \arrow[r] & C(g \circ f) \arrow[r] & C(g) \arrow[r] & C(f)[1]
\end{tikzcd}$$

$\exists$