

Derived categories Lecture 4

- 1) "Resolutions" are unique in the $K(\mathcal{A})$
- 2) The Homotopy category is triangulated

Recall \mathcal{A} abelian $\text{ob } K(\mathcal{A}) : \text{ob } \text{Kom}(\mathcal{A})$

$$\text{Hom}_{K(\mathcal{A})}(A, B) = \text{Hom}_{\text{Kom}(\mathcal{A})}(A, B)_0$$

Projective Resolutions \mathcal{A} has enough projectives

$$X \in \mathcal{A} \quad \exists \quad P^0 \text{ proj} \quad P^0 \xrightarrow{\epsilon_X} X$$

(K) Kernel of ϵ_X

$$P^{-1} \rightarrow K$$

$$P^{-1} \xrightarrow{\epsilon'} P^0 \xrightarrow{\epsilon_X} X$$

Reiterate

$$\cdots \rightarrow P^{-n} \rightarrow P^{-n+1} \rightarrow \cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow X$$

P^0 \downarrow
 0

$$H^i(P^\bullet) = 0 \quad \text{for } i \neq 0$$

$$H^0(P^\bullet) = X$$

P^\bullet is a projective resolution of X

a complex of projective objects P^i with a map

$$P^0 \xrightarrow{\epsilon_X} X \quad \text{such that induces}$$

isomorphisms in cohomology

Thm (Gelfand - Manin)

$$f: X \rightarrow Y \quad \text{morphism in } \mathcal{A}$$

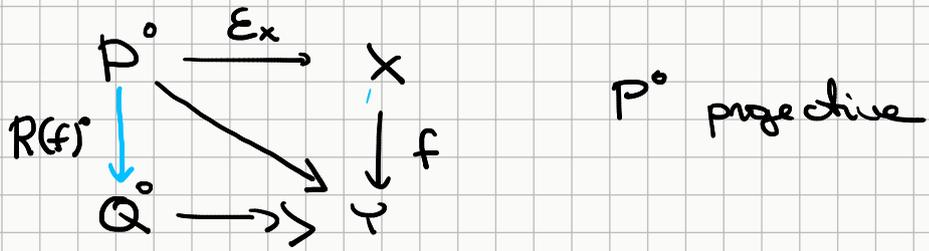
$$P^\bullet \xrightarrow{\epsilon_X} X \quad Q^\bullet \xrightarrow{\epsilon_Y} Y \quad \text{projective res.}$$

\exists a morphism $R(f): P^\bullet \rightarrow Q^\bullet$ such that

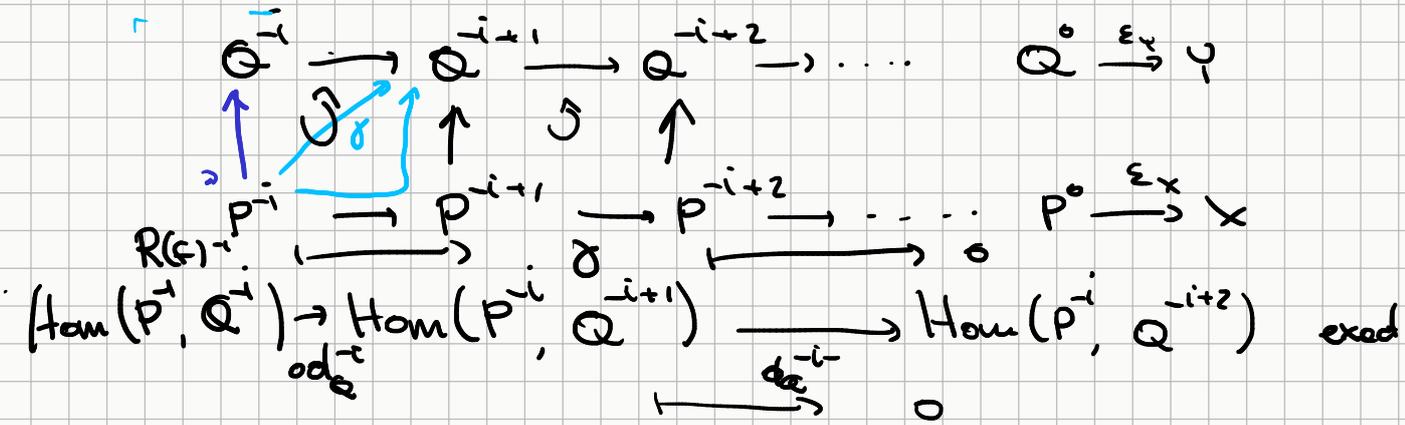
$$\begin{array}{ccc} P^0 & \xrightarrow{\epsilon_X} & X \\ R(f)^0 \downarrow & \circlearrowleft & \downarrow \\ Q^0 & \xrightarrow{\epsilon_Y} & Y \end{array}$$

$R(f)$ is unique up to homotopy equiv.

Proof:

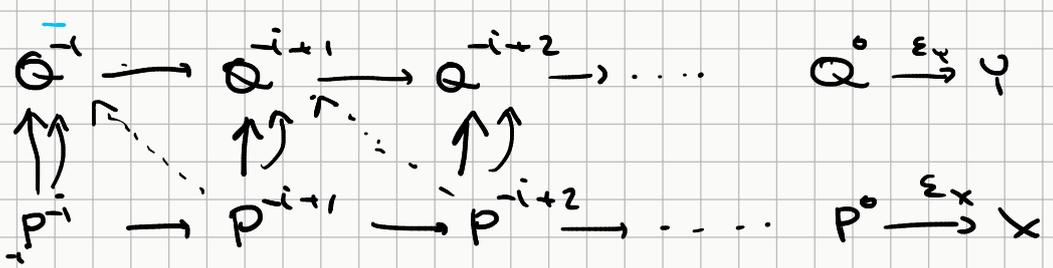


Inductive step: $\text{Hom}_X(P^{-i}, -)$ is exact if P^{-i} proj.

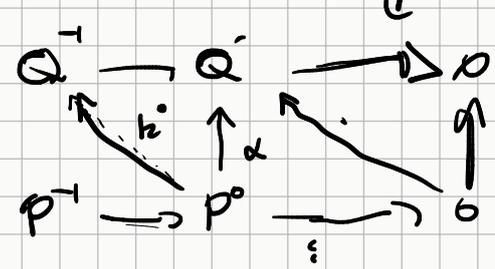


$\exists R(f)^{-i} \in \text{Hom}(P^{-i}, Q^{-i})$ such that

$$d_Q^{-i} R(f)^{-i} = \delta - R(f)^{-i+1} d_P^{-i}$$



$$\alpha = R(f)^0 - (R(f)^1)^0$$



epi + proj $\Rightarrow \exists k^0$

$$d_Q^{-1} k^0 + 0 d_P^0 = \alpha$$

$$\begin{array}{ccccccc}
 Q^{-i-1} & \longrightarrow & Q^{-i} & \longrightarrow & Q^{-i+1} & \longrightarrow & Q^{-i+2} \longrightarrow \dots & Q^0 \xrightarrow{\epsilon_X} U \\
 & & \uparrow & \nearrow & \uparrow & \nearrow & & \\
 P^{-i-1} & \longrightarrow & P^{-i} & \longrightarrow & P^{-i+1} & \longrightarrow & P^{-i+2} \longrightarrow \dots & P^0 \xrightarrow{\epsilon_X} X
 \end{array}$$

$$\alpha^{-i} = R(f)^{-i} - (R(f)')^{-i}$$

$$d_Q(\alpha^{-i} - k^{i+1} d_P) = 0$$

$$d_Q(\alpha^{-i}) = d^{i+1} d_P = (d_Q k^{i+1} + k^i d_P) d_P$$

$$= d_Q(k^{i+1} d_P)$$

$$\text{Hom}(P^{-i}, Q^{-i}) \xrightarrow{\gamma} \text{Hom}(P^{-i+1}, Q^{-i+1})$$

I can construct k^i as before.

Cor $f = \text{id}, X \rightarrow X$

Every "beck" resolution is homotopic equivalent to a projective one.

Given $P' \rightarrow Q'$ proj resolution of X

$$\begin{array}{ccc} & & \\ \epsilon_X \downarrow & X & \leftarrow \\ & & \exists R(\text{id}) : P' \rightarrow Q' \end{array}$$

The map induced in cohomology = id

$$X \in A \rightsquigarrow 0 \rightarrow 0 \rightarrow X \rightarrow 0 \rightarrow 0 \in K(A)$$

$$P' \rightarrow P^0 \rightarrow 0 \rightarrow 0$$

$$\downarrow 0 \quad \downarrow \epsilon_X \quad \leftarrow \text{map in } K(A)$$

$$0 \xrightarrow{0} X \rightarrow 0$$

Induces an isomorphism in cohomology.

Def $f: X^\bullet \rightarrow Y^\bullet$ is a quasi-isomorphism if induces an isomorphism in cohomology (at every level).

Mapping Cone (Kashiwara - Shapiro sheaves on Manifolds)

$$f: X^\bullet \rightarrow Y^\bullet \quad \text{in } \mathcal{K}(\mathcal{A})$$

The cone of f

$$\mathcal{K}(\mathcal{A}) \ni C(f) \xrightarrow{\cong} C(f)^n = X^{n+1} \oplus Y^n$$

$$\downarrow d_{C(f)} = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}$$

$$C(f)^{n+1} = X^{n+2} \oplus Y^{n+1}$$

$$\alpha(f): Y^\bullet \rightarrow C(f)$$

$$\alpha^n = \begin{pmatrix} 0 \\ \text{id}_{Y^n} \end{pmatrix}$$

$$\beta(f): C(f) \rightarrow X[\bullet]$$

$$\beta(f)^n = (\text{id}_{X^{n+1}} \ 0)$$

Lemma: For any morphism of complexes $f: X^\bullet \rightarrow Y^\bullet$

There is $\phi: X[\bullet] \rightarrow C(\alpha(f))$ inducing an homotopy equivalence (iso in $\mathcal{K}(\mathcal{A})$)

In addition we have the following commutative diagram in $\mathcal{K}(\mathcal{A})$

$$\begin{array}{ccccccc} Y & \xrightarrow{\alpha(f)} & C(f) & \xrightarrow{\beta(f)} & X[\bullet] & \xrightarrow{-f[\bullet]} & Y[\bullet] \\ \text{id} \downarrow & & \downarrow \text{id} & & \downarrow \phi & & \downarrow \text{id} \\ Y & \xrightarrow{\alpha(f)} & C(f) & \xrightarrow{\alpha(\alpha(f))} & C(\alpha(f)) & \xrightarrow{\quad} & Y[\bullet] \end{array}$$

A triangle (Δ) in $k(A)$ is a sequence of objects & maps

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow f & \searrow & \downarrow g & \searrow & \downarrow h & \searrow & \downarrow f[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

$$\begin{array}{ccc} & Z & \\ +1 \swarrow & & \searrow \\ X & \longrightarrow & Y \end{array}$$

A triangle is distinguished if it is isomorphic to

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} C(f) \xrightarrow{\beta(f)} X[1]$$

PROPERTIES:

TR 0 Any triangle isom to a distinguished one is distinguished

TR 2 Any morphism $f: X \rightarrow Y$ can be completed to a distinguished triangle

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ & & & & Z = C(f) & & \end{array}$$

TR 3

Rotation

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \quad \text{distinguished}$$

then

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{f[1]} Y[1] \quad \text{distinguished}$$

this is the lemma

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{\alpha(f')} & C(f') & \xrightarrow{\beta(f')} & X'[1] \end{array}$$

Lemma \Rightarrow

$$Y' \rightarrow C(f') \rightarrow X'[1] \rightarrow Y'[1] \quad \text{distinguished}$$

TR1 $X \xrightarrow{id} X \rightarrow 0 \rightarrow X[1]$ is distinguished

the cone $C(0 \rightarrow X) \cong X$
 $0 \oplus X^n$
 $\begin{pmatrix} 0 & 0 \\ 0 & d_{X,n} \end{pmatrix}$

$0 \rightarrow X \xrightarrow{id} X \rightarrow 0[1]$ is distinguished

ROTATE

$X \rightarrow X \rightarrow 0 \rightarrow X[1]$ distinguished

TR4 (5 lemma)

$$\begin{array}{ccccccc} X & \rightarrow & Y & \rightarrow & Z & \rightarrow & X[1] & \text{dist} \\ \downarrow u & \cong & \downarrow v & & \downarrow \cong & & \downarrow u[1] & \\ X' & \rightarrow & Y' & \rightarrow & Z' & \rightarrow & X'[1] & \end{array}$$

$$\begin{array}{ccc} Z = C(f) & \xrightarrow{u} & X^{n+1} \oplus Y^n \\ & & \downarrow \\ Z' & \xrightarrow{u'} & (X')^{n+1} \oplus (Y')^n \end{array}$$

$$\begin{pmatrix} u^n & 0 \\ s & v^n \end{pmatrix}$$

Given by the naturality of the diagonal commut

TRS (oh No!!!)

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \rightarrow & C(f) & \rightarrow & X[1] \\ \downarrow id & & \downarrow g & & \downarrow & & \downarrow id \\ X & \xrightarrow{g \circ f} & Z & \rightarrow & C(g \circ f) & \rightarrow & X[1] \\ \downarrow f & & \downarrow id & & \downarrow & & \downarrow f[1] \\ Y & \xrightarrow{g} & Z & \rightarrow & C(g) & \rightarrow & Y[1] \\ \downarrow \alpha(f) & & \downarrow \alpha(g \circ f) & & \downarrow \alpha(g) & & \downarrow \alpha(f)[1] \\ C(f) & \rightarrow & C(g \circ f) & \rightarrow & C(g) & \rightarrow & C(f)[1] \end{array}$$

□