Derived categories

Lecture 1 (first hour)

Jan 15, 2026

§ Motivation

'40, '50s   "Hom. alg." Cartan–Eilenberg '56   Tôhoku-paper '57

Homological algebra (Cartan, Eilenberg, Mac Lane, Grothendieck)

R (comm) ring $\rightsquigarrow$ Mod_R = category of R-modules (e.g. $\mathrm{Mod}_{\mathbb{Z}} = \mathrm{Ab}$)

- short exact sequences $\quad 0 \to K \to M \to N \to 0$
- chain complex $\quad \to M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \to \cdots \qquad d^2 = 0$

  cochain complex $\quad \to M^{-1} \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} \qquad d^2 = 0$
- homology $\quad H_i(M_\bullet) = \ker d_i / \operatorname{im} d_{i+1}$ measures non-exactness

  cohomology $\quad H^i(M^\bullet) = \ker d^i / \operatorname{im} d^{i-1}$
- derived functors

  Ex: $M \otimes_R -$ right-exact, derived functors $\mathrm{Tor}_i^R(M,N)$

  $\mathrm{Hom}_R(M,-)$ left-exact, —"— $\mathrm{Ext}_R^i(M,N)$

  $\mathrm{Hom}_R(-,N) : \mathrm{Mod}_R^{op} \to \mathrm{Mod}_R$ left-exact, —"—
- Calculated using proj/inj resolutions. / proj res

$$\cdots \xrightarrow{d_3} \underset{\text{proj}}{P_2} \xrightarrow{d_2} \underset{\text{proj}}{P_1} \xrightarrow{d_1} \underset{\text{proj}}{P_0} \to N \to 0 \quad \text{exact}$$

Def: $\mathrm{Tor}_i^R(M,N) := H_i(M \otimes_R P_\bullet)$,   independent of resolution

Rmk: $\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N$ (by right-exactness of $\otimes$)

Example

Ex: $M = \mathbb{Z}/p\mathbb{Z}$, $N = \mathbb{Z}/n\mathbb{Z}$

$$0 \to \underbrace{\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}}_{P_\bullet} \to \underbrace{\mathbb{Z}/n\mathbb{Z}}_{N} \to 0$$

$$\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} P_\bullet = \mathbb{Z}/p\mathbb{Z} \xrightarrow[d_1]{\cdot n} \mathbb{Z}/p\mathbb{Z}$$

$$\mathrm{Tor}_i^R(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \frac{\ker(d_i)}{\mathrm{im}(d_{i+1})} = \begin{cases} \mathrm{coker}(d_1) & \text{if } i=0 \\ \ker(d_1) & \text{if } i=1 \\ 0 & \text{o/w} \end{cases}$$

$p \nmid n$: $\mathbb{Z}/p\mathbb{Z} \xrightarrow[\cong]{\cdot n} \mathbb{Z}/p\mathbb{Z}$ $\quad \mathrm{Tor}_0 = \mathrm{Tor}_1 = 0$

$p \mid n$: $\mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \mathbb{Z}/p\mathbb{Z}$ $\quad \mathrm{Tor}_0 = \mathrm{Tor}_1 = \mathbb{Z}/p\mathbb{Z}$

Ex: $R = \mathbb{Z}$, $\quad 0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ SES

$\rightsquigarrow$ LES:

$\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} -$:

$$\underset{\substack{\parallel \\ 0}}{\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z})} \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/(n,p)\mathbb{Z} \to 0$$

$p \nmid n$:

$$0 \to 0 \to \mathbb{Z}/p \xrightarrow[\cong]{\cdot n} \mathbb{Z}/p\mathbb{Z} \to 0 \to 0$$

$p \mid n$

$$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/p\mathbb{Z} \to 0$$

$\boxed{\text{Observation}}$ Not only is $H_i(M \otimes P_\bullet)$ indep. of $P_\bullet$ but the chain complex $M \otimes P_\bullet$ is indep. <u>up to quasi-isom</u>:

<u>Def</u>: $\varphi : C_\bullet \longrightarrow D_\bullet$ quasi-isomorphism if $H_i(\varphi)$ iso. $\forall i$.

<u>Ex</u>: $\to P_2 \to P_1 \to P_0 \to N$ proj resolution

$\Rightarrow \quad P_\bullet \longrightarrow N[0]$ quasi-iso

$$\begin{tikzcd} \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[d] \\ \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & N \end{tikzcd}$$

$$H_i(P_\bullet) = \begin{cases} N & \text{if } i=0 \\ 0 & \text{o/w} \end{cases}$$

$$H_i(N[0]) = \text{—}\,''\,\text{—}$$

<u>Ex</u>: Given projective resolutions $P_\bullet \longrightarrow N$ and $P_\bullet' \longrightarrow N$

$\exists\ P_\bullet \xrightarrow{\varphi} P_\bullet'$ compatible w/ N.

$$H_i(P_\bullet) \longrightarrow H_i(P_\bullet') \qquad \begin{matrix} 0 \longrightarrow 0 & \text{if } i \neq 0 \\ N \xrightarrow{\simeq} N & \text{if } i = 0 \end{matrix}$$

so φ quasi-iso.

<u>Derived category</u> (Grothendieck, Verdier 1963)

$\boxed{\text{Idea}}$ $D(R)$ = chain complexes up to quasi-isom.

$$\begin{aligned} \mathrm{Mod}_R &\longrightarrow D(R) \\ N &\longmapsto N[0] \end{aligned} \qquad P_\bullet \simeq N[0] \text{ in } D(R)$$

Now $M \overset{\mathbb{L}}{\underset{R}{\otimes}} N := M \underset{R}{\otimes} P_\bullet$ well-defined in $D(R)$.

$$\mathbb{Z}/n\mathbb{Z}[0] \cong \left(\underset{1}{\mathbb{Z}} \xrightarrow{\cdot n} \underset{0}{\mathbb{Z}} \right) \quad \text{in } D(\mathbb{Z})$$

Why derived categories?

① Natural setting for homological algebra
(derived functors, spectral sequences, ...)

② Duality theorems
- "Usual duality"
- Grothendieck duality (coherent sheaves)
- Verdier duality (torsion/ℓ-adic sheaves)

③ Many natural objects in $D(R)$ rather than Mod_R

- Dualizing complexes
- Cotangent complex
- Perverse sheaves
- perfect complexes (cf. vector bundles)

④ Many natural functors only on derived level, e.g.,

- 6FF $\quad \overset{\mathbb{L}}{\otimes}$, $\mathcal{H}om$, Lf^*, Rf_*, $Rf_!$, $f^!$
- Fourier–Mukai

⑤ $D(R)$ interesting invariant of R. (instead of Mod_R)

- birational vs derived equiv etc.

⑥ Moduli of obj in $D(R)$ (instead of Mod_R)

- Bridgeland stability
- Stable pairs

<u>Ex of ②</u>: Duality

$\boxed{\text{Vector spaces}}$ Contravariant functor

$$\mathrm{Vect}_k^{op} \longrightarrow \mathrm{Vect}_k$$
$$V \longmapsto V^* := \mathrm{Hom}_k(V, k)$$

Natural map $V \longrightarrow (V^*)^*$ isomorphism if V fin. dim.

$\boxed{\text{Modules}}$ $R = \mathbb{Z}$

$$Ab^{op} \longrightarrow Ab$$
$$M \longmapsto M^* := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$$

Natural map $V \longrightarrow (V^*)^*$ isomorphism if M <u>free of finite rank</u>
($\Leftarrow$ f.g. projective)

<u>Ex</u>: $(\mathbb{Z}/n\mathbb{Z})^* = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$.

so $\mathbb{Z}/n\mathbb{Z} \xrightarrow{\not\cong} (\mathbb{Z}/n\mathbb{Z})^{**}$

Derived:

$$\mathbb{R}\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) := \mathbb{R}\mathrm{Hom}(\underset{1}{\mathbb{Z}} \xrightarrow{\cdot n} \underset{0}{\mathbb{Z}}, \mathbb{Z})$$

$$= \underset{0}{\mathbb{Z}^*} \xrightarrow{\cdot n} \underset{-1}{\mathbb{Z}^*} \cong \underset{0}{\mathbb{Z}} \xrightarrow{\cdot n} \underset{-1}{\mathbb{Z}} \simeq \mathbb{Z}/n\mathbb{Z}[-1]$$

$$\mathbb{R}\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}[-1], \mathbb{Z}) = \cdots = \mathbb{Z}/n\mathbb{Z}[0]$$

Let $\mathbb{D}(-) = \mathbb{R}\mathrm{Hom}(-, R) : D(R) \to D(R)$.

Then $\exists\, M \to \mathbb{D}(\mathbb{D}(M))$ which is iso on <u>perfect complexes</u>

those quasi-iso to $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0$

w/ P_i proj, f.g.