Derived categories

Lecture 1 (second hour)

Jan 15, 2026

Category $\mathcal{C}$:

- set $\mathrm{ob}\,\mathcal{C}$
- $\mathrm{Hom}_{\mathcal{C}}(A,B)$, $A,B \in \mathcal{C}$ such that
  - $1_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$ identity
  - $\mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C)$ composition

Ex: Grp, DiffMfld, Top, Mod, Set

Ex: $\mathcal{C}^{op}$: $\mathrm{ob}\,\mathcal{C}^{op} = \mathrm{ob}\,\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}^{op}}(A,B) := \mathrm{Hom}_{\mathcal{C}}(B,A)$$

(covariant)
Functor b/w two categories: $F: \mathcal{C} \longrightarrow \mathcal{D}$ is given by

$$A \longmapsto F(A)$$
$$A \xrightarrow{f} B \longmapsto F(A) \xrightarrow{F(f)} F(B)$$

s.th. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$, $F(g) \circ F(f) = F(g \circ f)$

Contravariant functor $\mathcal{C} \to \mathcal{D}$ is $F: \mathcal{C}^{op} \longrightarrow \mathcal{D}$

Two categories are **isomorphic** if $\exists\, F: \mathcal{C} \longrightarrow \mathcal{D}$ s.th. $F \circ G = \mathrm{id}_{\mathcal{D}}$
$G: \mathcal{D} \longrightarrow \mathcal{C}$ $G \circ F = \mathrm{id}_{\mathcal{C}}$

Ex: $\mathrm{Ab} \simeq \mathrm{Mod}_{\mathbb{Z}}$

A natural transformation (morphism of functors) $\eta : F \to G$ b/w functors $F, G : \mathcal{C} \to \mathcal{D}$

$$\begin{array}{c|c} \mathcal{C} & \mathcal{D} \\ \begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} & \begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circ"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd} \end{array}$$

natural iso if η_A iso $\forall A$

A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if $\exists\, G : \mathcal{D} \to \mathcal{C}$ s.th. $\exists$ $F \circ G \simeq 1_{\mathcal{D}}$ natural isomorphisms
$G \circ F \simeq 1_{\mathcal{C}}$

Ex: $\underline{\mathbb{N}} \longrightarrow Vect^{fr}$ equiv ob $\underline{\mathbb{N}} = \mathbb{N}$
$n \longmapsto (k^n, e_1, \dots, e_n)$ $Hom_{\underline{\mathbb{N}}}(m, n) = Mat_{n \times m}$

Def: A functor $F : \mathcal{C} \to \mathcal{D}$ is

- faithful if $Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{D}}(F(A), F(B))$ inj
- full ——— " ——— surj
- fully faithful ——— " ——— bij
- ess surj if $\forall M \in \mathcal{D}$ $\exists N \in \mathcal{C}$ s.th. $F(N) \simeq M$

($A \simeq B$ if $\exists\, \varphi : A \to B$ & $\psi : B \to A$ s.th. $\varphi \circ \psi = id_B$, $\psi \circ \varphi = id_A$)

Thm: F is an equivalence $\Longleftrightarrow$ F ff. + ess surj

Linear categories

R ring. A category $\mathcal{C}$ is **R-linear** (linear if $R = \mathbb{Z}$)
if $\forall A, B \in \mathcal{C}$

- $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is an R-module (ab. grp)
- composition is R-bilinear

A zero object in a category $\mathcal{C}$ is an object $0 \in \mathrm{ob}\,\mathcal{C}$ s.th:

$$\forall M \in \mathcal{C} \quad \exists!\ 0 \in \mathrm{Hom}_{\mathcal{C}}(0, M) \quad \text{(initial)}$$
$$\exists!\ 0 \in \mathrm{Hom}_{\mathcal{C}}(M, 0) \quad \text{(final)}$$

(If $\mathcal{C}$ R-linear: M zero obj $\Leftrightarrow$ $\mathrm{Hom}(M,M) = 0$.)