Derived categories

Lecture 5

Feb 19, 2026

§ Quasi-isomorphisms

Def: A morphism of complexes $A^\bullet \xrightarrow{f} B^\bullet$ is a ***quasi-isomorphism*** if $H^i(f): H^i(A^\bullet) \to H^i(B^\bullet)$ isomorphism $\forall i$.

Ex: $\to P^{-1} \to P^0 \to A \to 0$ projective resolution
Then $P^\bullet \to A[0]$ qiso

Ex: $0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$ inj resolution
Then $A[0] \to I^\bullet$ qiso.

Def: $C^\bullet$ ***acyclic*** if $H^i(C^\bullet) = 0$ $\forall i$.

Ex: $0 \to A^\bullet \xrightarrow{f} B^\bullet \to C(f) \to 0$
Then f qiso $\iff$ $C(f)$ acyclic.

Ex: $C^\bullet$ acyclic $\iff$ $0 \to C^\bullet$ qiso $\iff$ $C^\bullet \to 0$ qiso.

Ex: $0 \to A \xrightarrow{f} B \to B/A \to 0$ $\rightsquigarrow$ $C(f) \to B/A[0]$ qiso

$$C(f) = \big(\underset{-1}{A} \to \underset{0}{B}\big)$$

Warning: For $A^\bullet$ and $B^\bullet$ to be qiso not enough that $H^i(A^\bullet)$ and $H^i(B^\bullet)$ iso $\forall i$. Need a morphism of complexes $f: A^\bullet \to B^\bullet$ inducing isos $H^i(f)$.

Rmk: $f, g: A^\bullet \longrightarrow B^\bullet$. If $f \sim g$ then $H^i(f) = H^i(g)$

$\Rightarrow$ If f iso in $K(\mathcal{A})$, i.e., $\exists f^{-1}$ s.th. $f \circ f^{-1} \sim \mathrm{id}$
$f^{-1} \circ f \sim \mathrm{id}$

then f qiso.

Def: Truncations: $\left(\tau^{\leq a} A^\bullet\right)^i = \begin{cases} A^i & \text{if } i < a \\ \ker(d^i) & \text{if } i = a \\ 0 & \text{if } i > a \end{cases}$

$$\begin{tikzcd}
A^\bullet: & \cdots \arrow[r] & A^{-1} \arrow[r] & A^0 \arrow[r, "d^0"] & A^1 \arrow[r, "d^1"] & A^2 \arrow[r, "d^2"] & A^3 \arrow[r] & \cdots \\
\tau^{\leq 1} A^\bullet: & & A^{-1} \arrow[r] \arrow[u, equal] & A^0 \arrow[r] \arrow[u, equal] & \ker(d^1) \arrow[r] \arrow[u, hook] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

so $\tau^{\leq a} A^\bullet \xrightarrow{\alpha} A^\bullet$ and $H^i(\alpha) = \begin{cases} \text{iso} & \forall i \leq a \\ 0 & \forall i > a \end{cases}$

$\left(\tau^{\geq a} A^\bullet\right)^i = \begin{cases} 0 & \text{if } i < a \\ \mathrm{coker}(d^{i-1}) & \text{if } i = a \\ A^i & \text{if } i > a \end{cases}$

$$\begin{tikzcd}
A^{a-1} \arrow[r] \arrow[d] & A^a \arrow[r] \arrow[d, two heads] & A^{a+1} \arrow[d, equal] \\
0 \arrow[r] & \mathrm{coker}(d^{i-1}) \arrow[r] & A^{a+1}
\end{tikzcd}$$

so $A^\bullet \longrightarrow \tau^{\geq a} A^\bullet$

§ Derived category

Idea: $\mathcal{A}$ abelian category $\rightsquigarrow \mathrm{Kom}(A)$

$$\rightsquigarrow K(A) = \mathrm{Kom}(A)/_{htp} \rightsquigarrow D(A) = K(A)/_{qiso}$$

Thm (Huybrechts 2.10, Gelfand-Manin III.2) Let $\mathcal{A}$ abel. cat.
$\exists$ triangulated category $D(A)$ and functor $Q: \mathrm{Kom}(A) \to D(A)$ s.th.

(i) $Q(\text{qiso}) = \text{iso}$

(ii) If D category and $F: \mathrm{Kom}(\mathcal{A}) \to D$ functor s.th. $F(\text{qiso}) = \text{iso}$, then F factors uniquely through Q.

$$\begin{tikzcd} \mathrm{Kom}(\mathcal{A}) \arrow[r, "Q"] \arrow[dr, "F"'] & D(A) \arrow[d, dashed, "\exists ! G"] \\ & D \end{tikzcd}$$

s.th. $F = G \circ Q$

1st proof of existence:

Let • $\mathrm{Ob}\, D(A) = \mathrm{Ob}\, \mathrm{Kom}(A)$

(zigzags)

• morphisms $A^\bullet \to B^\bullet$ in $D(A)$ be finite "paths" consisting of usual morphisms and inverses of quasi-isomorphisms:

$$A^\bullet \xrightarrow{f_1} A_1^\bullet \xrightarrow{f_2} A_2^\bullet \overset{\text{qiso}}{\underset{s_1}{\longleftarrow}} A_3^\bullet \xrightarrow{f_3} A_4^\bullet \overset{\text{qiso}}{\underset{s_2}{\longleftarrow}} A_5^\bullet \overset{\text{qiso}}{\underset{s_3}{\longleftarrow}} A_6^\bullet \xrightarrow{f_4} B^\bullet$$

$$f = f_4\, s_3^{-1}\, s_2^{-1}\, f_3\, s_1^{-1}\, f_2\, f_1$$

+ equivalence relation $\xrightarrow{f_1}\xrightarrow{f_2} \;=\; \xrightarrow{f_2 \circ f_1}$

$$\xrightarrow{s} \overset{s}{\underset{\text{qiso}}{\longleftarrow}} \;=\; \xrightarrow{id} \;=\; \overset{s}{\underset{\text{qiso}}{\longleftarrow}} \xrightarrow{s}$$

Ex: In $\mathrm{Kom}(\mathcal{A})$:

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow C(f) \xrightarrow{\delta} A^\bullet[1] \longrightarrow B^\bullet[1] \longrightarrow \cdots$$

$$\Rightarrow \quad H^0(A^\bullet) \longrightarrow H^0(B^\bullet) \longrightarrow H^0(C(f)) \xrightarrow{\delta} H^1(A^\bullet) \longrightarrow H^1(B^\bullet) \longrightarrow \cdots$$

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow B^\bullet/A^\bullet \longrightarrow 0$$

In general, $\nexists\ B^\bullet/A^\bullet \xrightarrow{\delta} A^\bullet[1]$ inducing $H^0(B^\bullet/A^\bullet) \longrightarrow H^1(A^\bullet)$

but

$$\begin{tikzcd}
 & C(f) \arrow[dl, "\text{qiso}"'] \arrow[dr] & \\
B^\bullet/A^\bullet & & A^\bullet[1]
\end{tikzcd}$$

so have $B^\bullet/A^\bullet \longrightarrow A^\bullet[1]$ in $D(\mathcal{A})$ inducing

$$\begin{tikzcd}
 & H^0(C(f)) \arrow[dl, "\cong"'] \arrow[dr, "\delta"] & \\
H^0(B^\bullet/A^\bullet) \arrow[rr] & & H^1(A^\bullet[1])
\end{tikzcd}$$

Drawback: long paths difficult to work with. Would like roofs:

$$\mathrm{Hom}_{D(\mathcal{A})}(A_\bullet, B_\bullet) = \left\{ \begin{tikzcd}
 & C_\bullet \arrow[dl, "\text{qiso}"', "s"] \arrow[dr, "f"] & \\
A_\bullet & & B_\bullet
\end{tikzcd} \right\} \Big/ \sim$$

where $fs^{-1} \sim gt^{-1}$ if $\exists$

$$\begin{tikzcd}
 & & D_\bullet \arrow[dl, "u"] \arrow[dr] \arrow[ddll, bend right, "\text{qiso}"'] & & \\
 & C_\bullet \arrow[dl, "\text{qiso}"', "s"] \arrow[drr, "f"] & & C_\bullet' \arrow[dr, "g"] \arrow[dlll, "\text{qiso}", "t"'] & \\
A_\bullet & & & & B_\bullet
\end{tikzcd}$$

For this to work need that qiso's form a **left mult. system**:

(a) s_1, s_2 qiso $\Rightarrow$ $s_1 \circ s_2$ qiso   ok!

(b) Given

$$\begin{tikzcd} C^\bullet \arrow[dr, "f"'] & & E^\bullet \arrow[dl, "s"', "\text{qiso}"] \\ & D^\bullet & \end{tikzcd}$$

$\exists$

$$\begin{tikzcd} & F^\bullet \arrow[dl, "t", "\text{qiso}"'] \arrow[dr] & \\ C^\bullet \arrow[dr, "f"'] \arrow[rr, phantom, "\circ"] & & E^\bullet \arrow[dl, "s"] \\ & D^\bullet & \end{tikzcd}$$

(c) If $C^\bullet \underset{g}{\overset{f}{\rightrightarrows}} D^\bullet \xrightarrow[\text{qiso}]{s} E^\bullet$ s.th. $sf = sg$

then $\exists\ F^\bullet \xrightarrow[\text{qiso}]{t} C^\bullet \underset{g}{\overset{f}{\rightrightarrows}} D^\bullet$ s.th. $ft = gt$

Using (b) we can compose roofs. Using (c) it follows that composition is unique (up to the equiv. $\sim$)

$$\begin{tikzcd} & & E^\bullet \arrow[dl, "u", "\text{qiso}"'] \arrow[dr, "h"] & & \\ & B^\bullet \arrow[dl, "s", "\text{qiso}"'] \arrow[dr, "f"] & & D^\bullet \arrow[dl, "t", "\text{qiso}"'] \arrow[dr, "s"] & \\ A^\bullet & & C^\bullet & & E^\bullet \end{tikzcd}$$

$$(g t^{-1}) \circ (f s^{-1}) := (gh) \circ (su)^{-1}$$

(b) + (c) also used to prove that $\sim$ is an equiv. relation.

Prop (GM III.5.2): $\mathcal{A} \longrightarrow D(\mathcal{A})$ is fully faithful
$A \longmapsto A[0]$

pf: $\mathrm{Hom}_{\mathcal{A}}(A,B) \longrightarrow \mathrm{Hom}_{D(A)}(A[0], B[0])$

RHS:

$$\begin{tikzcd}
 & C^\bullet \arrow[dl, "s", "\text{qiso}"'] \arrow[dr, "f"] & \\
A[0] & & B[0]
\end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd}
 & C^\bullet \arrow[ddl, "\text{qiso}"'] \arrow[d, "\text{qiso}"] \arrow[ddr] & \tau^{\leq 0}\tau^{\geq 0}C^\bullet = A[0] \arrow[dl, "\text{qiso}"'] \arrow[dd, "g"] \\
 & \tau^{\geq 0}C^\bullet \arrow[dl] \arrow[dr] & \\
A[0] = \tau^{\geq 0}A[0] & & \tau^{\geq 0}B[0] = B[0]
\end{tikzcd}$$

So $fs^{-1} \sim g\,\mathrm{id}^{-1}$ $\square$

§ Sign conventions

Recall: $A^\bullet \xrightarrow{f} B^\bullet \longrightarrow C(f) \longrightarrow A^\bullet[1]$

Shift: $(A^\bullet[1])^i = A^{i+1}$, $\quad d^i_{A^\bullet[1]} = -d_A^{i+1}$

$f[1]^i = f^{i+1}$

Cone $\quad C(f) = A^\bullet[1] \oplus B^\bullet$, $\quad d_{C(f)} = \begin{pmatrix} d_{A^\bullet[1]} & 0 \\ f[1] & d_{B^\bullet} \end{pmatrix}$

i.e., $\quad C(f)^i = A^{i+1} \oplus B^i \quad d^i_{C(f)} = \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d^i_B \end{pmatrix}$

TR 3: $\quad X \xrightarrow{f} Y \xrightarrow{\tau} C(f) \xrightarrow{\pi} X[1]$

$\rightsquigarrow \quad Y \xrightarrow{\tau} C(f) \xrightarrow{\pi} X[1] \xrightarrow{-f[1]} Y[1]$

§ Proof that qiso's LMS

Warning: qiso's in $\mathrm{Kom}(\mathcal{A})$ <u>not</u> LMS. Need to work in $K(\mathcal{A})$.

Prop: (Huybrechts Prop 2.17, G-M III.4) qiso in $K(\mathcal{A})$ LMS.

pf: (b)

$$\begin{tikzcd}
 & \exists\ C_0^\bullet \arrow[dl, "t"', "\text{qiso}"] \arrow[dr, "h"] & \\
C^\bullet \arrow[dr, "g"'] & & A^\bullet \arrow[dl, "s", "\text{qiso}"'] \\
 & B^\bullet &
\end{tikzcd}$$

$$\begin{tikzcd}
C_0 = C(\tau g)[-1] \arrow[r, "t"] \arrow[d, "h"'] & C^\bullet \arrow[r, "\tau g"] \arrow[d, "g"] & C(s) \arrow[r] \arrow[d, equal] & C(\tau_g) \arrow[d, "③"] \arrow[dd, dashed, bend right, "②"'] & \\
A^\bullet \arrow[r, "s"] & B^\bullet \arrow[r, "\tau"] \arrow[d, equal] & C(s) \arrow[r] \arrow[d, equal] & A^\bullet[1] \arrow[r] \arrow[d, "①"', "\cong"] & B^\bullet[1] \arrow[d, equal] \\
 & B^\bullet \arrow[r, "\tau"] & C(s) \arrow[r] & C(\tau) \arrow[r] & B^\bullet[1]
\end{tikzcd}$$

① Prop 2.16: $\exists$ <u>isomorphism</u> in $K(\mathcal{A})$

② $\exists$ by construction of cone:
$$C(\tau_g) = C^\bullet[1] \oplus C(s) \xrightarrow{(g,\, id)} C(\tau) = B^\bullet[1] \oplus C(s)$$

③ by ① + ②

Finally, s qiso $\Rightarrow$ $C(s)$ acyclic $\Rightarrow$ t qiso $\square$

In the proof we used Prop 2.16:

Prop 2.16: (TR3 for $K(\mathcal{A})$) [Sofia did last time]

Consider $A^\bullet \xrightarrow{f} B^\bullet$ and mapping cones

$$\begin{tikzcd}
A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "\tau"] \arrow[d, equal] & C(f) \arrow[r, "\pi"] \arrow[d, equal] & A^\bullet[1] \arrow[r, "-f[1]"] \arrow[d, dashed, "\cong", "g"'] & B^\bullet[1] \arrow[d, equal] \\
& B^\bullet \arrow[r, "\tau"] & C(f) \arrow[r] & C(\tau) \arrow[r] & B^\bullet[1]
\end{tikzcd}$$

Then $\exists$ isom. g in $K(\mathcal{A})$ making diagram commutative.

(so rotation $B^\bullet \to C(f) \to A^\bullet[1] \to B^\bullet[1]$ is dist.)

pf:

$$\begin{tikzcd}
C(f) = A^\bullet[1] \oplus B^\bullet \arrow[r, "pr_1 = \pi"] \arrow[d, equal] & A^\bullet[1] \arrow[r, "-f[1]"] \arrow[d, "g = (-f[1], id, 0)"] & B^\bullet[1] \arrow[d, equal] \\
C(f) = A^\bullet[1] \oplus B^\bullet \arrow[r, "{(0, pr_1, pr_2)}"] & B^\bullet[1] \oplus A^\bullet[1] \oplus B^\bullet \arrow[r, "pr_1"] & B^\bullet[1]
\end{tikzcd}$$

$B^\bullet[1] \oplus A^\bullet[1] \oplus B^\bullet = C(\tau)$

$$d_{C(f)} = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} \qquad d_{C(\tau)} = \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ id & f & d_B \end{pmatrix}$$

- right square commutes

- $g^{-1} = pr_2$ is inverse in $K(A)$ b/c

$$g^{-1} \circ g = id,$$
$$\underset{\substack{\shortparallel \\ (-f \circ pr_2,\, pr_2,\, 0)}}{g \circ s^{-1}} = id + dh + hd \quad \text{where } h = (pr_3, 0, 0)$$

$$(g \circ s^{-1})(x,y,z) = (-f(y), y, 0)$$

$$d(x,y,z) = (-dx, -dy, x + f(y) + dz)$$

$$dh(x,y,z) = (-dz, \quad 0 \quad, z)$$
$$hd(x,y,z) = (x + f(y) + dz, 0, 0)$$

- left square commutes in $K(A)$ b/c

$$g^{-1} \circ (0, pr_1, pr_2) = pr_1 \qquad \square$$