

§1 Bounded versions

Def: $Kom^-(\mathcal{A}) = \{A^\bullet : A^i = 0 \ \forall i \gg 0\}$ "bounded below"
 $Kom^+(\mathcal{A}) = \{A^\bullet : A^i = 0 \ \forall i \ll 0\}$ "bounded above"
 $Kom^b(\mathcal{A}) = \{A^\bullet : \forall |i| \gg 0\}$ "bounded"
 $= Kom^- \cap Kom^+$

$$K^*(\mathcal{A}) = Kom^*(\mathcal{A}) / \sim \quad * \in \{+, -, b\}$$

$$D^*(\mathcal{A}) = K^*(\mathcal{A}) / q_{iso}$$

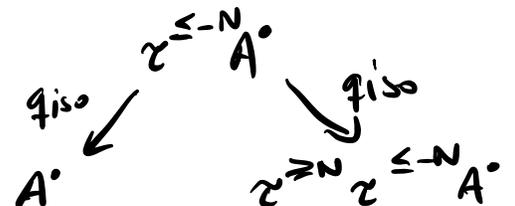
Prop H 2.30: $D^*(\mathcal{A}) \xrightarrow{\phi} D(\mathcal{A})$ fully faithful w/ essential image

$\{A^\bullet : H^i(A^\bullet) = 0 \ \forall i \gg 0\}$ "coh. bounded below"
 $\forall i \ll 0$ "coh bounded above"
 $\forall |i| \gg 0$ "coh bounded"

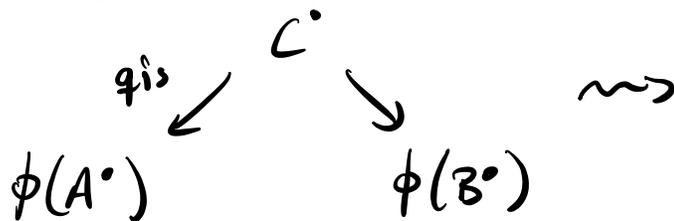
pf: Suppose $H^i(A^\bullet) = 0 \ \forall i > N$. Then $\tau^{\leq N} A^\bullet \rightarrow A^\bullet$ qiso
 so $A^\bullet \cong \phi(\tau^{\leq N} A^\bullet)$ in ess image of ϕ .

Similarly if $H^i(A^\bullet) = 0 \ \forall i < -N$, then $A^\bullet \rightarrow \tau^{\geq -N} A^\bullet$ qiso.
 etc

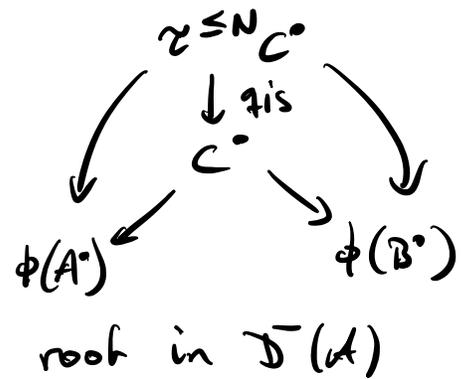
Similarly if $H^i(A^\bullet) = 0 \ \forall |i| > N$, then
 etc



For fully faithful: if $A^i = B^i = 0 \forall i > N$



etc



□

Rank: Not true in $K(A)$.

Rank: Same arg. as $A \rightarrow D(A)$ fully faithful prev. lecture.

Step 1:

$$\begin{array}{ccccc}
 A^0 & \xrightarrow{d^0} & A^1 & & \\
 f^0 \downarrow & & 0 \downarrow & & \\
 I^0 & \longrightarrow & I^0 \amalg_{A^0} A^1 & \xrightarrow{\text{inj}} & I^1
 \end{array}$$

$$\text{coker} \left(A^0 \xrightarrow{\begin{matrix} \text{1, 2, } d^0 - \text{1, } f^0 \\ \parallel \\ A^0 \end{matrix}} I^0 \oplus A^1 \right) = I^0 \oplus A^1 / \langle (-f^0(a), d^0(a)) : a \in A^0 \rangle$$

gives

$$\begin{array}{ccccccc}
 A^0 & \xrightarrow{d^0} & A^1 & \longrightarrow & A^2 & \longrightarrow & A^3 \longrightarrow \dots \\
 f^0 \downarrow & & 0 \downarrow & & f^1 \downarrow & & \downarrow & & \downarrow \\
 I^0 & \xrightarrow{d_I^0} & I^1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
 \end{array}$$

Claim: $H^0(f)$ isomorphism
 $H^1(f)$ injective.

pl:

$$\begin{array}{ccc}
 H^0(A^0) & \xrightarrow{H^0(f)} & H^0(I^0) \\
 \parallel & & \parallel \\
 \ker d^0 & \xrightarrow{\tilde{f}^0} & \ker d_{I^0} \\
 \searrow f^0 & & \cap \\
 & & I^0
 \end{array}$$

- \tilde{f}^0 inj. b/c f^0 inj

- \tilde{f}^0 surj b/c

$$i^0 \in \ker d_{I^0} \Leftrightarrow (i^0, 0) = (0, 0) \text{ in } I^0 \amalg_{A^0} A^1$$

$$\Leftrightarrow \exists a^0 \in A^0 \text{ s.t. } f^0(a^0) = i^0, d^0(a^0) = 0$$

(Here I use elements)

$$\Leftrightarrow i^0 \in f^0(\ker d^0).$$

$$\bullet \quad \begin{array}{ccc} H^i(A^\bullet) & \xrightarrow{H^i(f^\bullet)} & H^i(I^\bullet) \\ \parallel & & \parallel \\ \ker d^i / \text{im } d^i & \longrightarrow & I^i / d_I^i(I^{i-1}) \end{array}$$

$$\bar{a} \in A^i / d^i(A^{i-1}) \xrightarrow{\bar{f}^i}$$

(using elements)

$$\begin{aligned} \bar{f}^i \text{ injective} \quad \text{b/c: } f^i(a) = d_I^i(i) \text{ for some } i \in I^i \\ \Leftrightarrow (0, a) \sim (i, 0) \text{ --- " ---} \\ \Leftrightarrow \exists a' \in A^{i-1} : a = d^i(a'), \quad i = f^i(a') \\ \Leftrightarrow a \in \text{im } d^i \quad \blacksquare \end{aligned}$$

Step $i+1$ Suppose

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^0 & \longrightarrow & \dots & \longrightarrow & A^i & \longrightarrow & A^{i+1} \\ & & f^0 \downarrow & & & & f^i \downarrow & & \downarrow \\ 0 & \longrightarrow & I^0 & \longrightarrow & \dots & \longrightarrow & I^i & \longrightarrow & 0 \end{array}$$

$H^j(f^\bullet)$ iso $\forall j < i$ and inj for $j = i$

$$\begin{array}{ccccccc} A^{i-1} & \xrightarrow{d^{i-1}} & A^i & \xrightarrow{d^i} & A^{i+1} & & \\ f^{i-1} \downarrow & & \downarrow f^i & & \searrow f^{i+1} & & \\ I^{i-1} & \xrightarrow{d_I^{i-1}} & I^i & \longrightarrow & \text{coker } d_I^{i-1} & \longrightarrow & \text{coker } d_I^i \oplus A^{i+1} & \hookrightarrow & I^{i+1} \end{array}$$

$H^i(f^\bullet)$ still injective.
 $H^i(f^\bullet)$ surjective by cons.
 $H^{i+1}(f^\bullet)$ injective ---
 $d_I^i \circ d_I^{i-1} = 0$ by cons

so now in assumption for Step $i+2$.

Step ∞ :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \dots \\
 & & f^0 \downarrow & & \downarrow f^1 & & \\
 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots
 \end{array}$$

$$A^0 \longrightarrow I^0 \text{ is iso.}$$

Cor H 236: If $A^0 \in D(\mathcal{A})$ coh. left banded
 then $\exists A^0 \rightarrow I^0$ is iso.

pf: $A^0 \xrightarrow{\text{qis}} \mathcal{Z}^{\geq -N} A^0 \xrightarrow{\text{qis}} I^0$

D

Remark: We only used that $\forall A \in \mathcal{A}, \exists I \hookrightarrow A, I \in \mathcal{I} \subset \mathcal{A}$
 We never used that $I \in \mathcal{I}$ are injective.
 Same argument thus works for other classes $\mathcal{I} \subset \mathcal{A}$.

§3 $D^+(A) = K^+(\mathcal{I})$

Def: $\mathcal{I} = \{I \in A : I \text{ inj}\} \subset A$ full additive subcategory.

Rmk: \mathcal{I} additive, not abelian. (0 $\in \mathcal{I}$ and $(I, J \in \mathcal{I} \Rightarrow I \oplus J \in \mathcal{I})$)

Def: $K^*(\mathcal{I}) = \text{Kom}^*(\mathcal{I}) / \sim$ $* \in \{-, t, b\}$

triangulated category: dist triangles are triangles iso to cone triangles

$$A \xrightarrow{f} B \xrightarrow{g} C(f) \xrightarrow{\pi} A[1] \quad \text{w/ } A, B \in \text{Kom}(\mathcal{I}) \\ \Rightarrow C(f) \in \text{Kom}(\mathcal{I})$$

Prop A.2.40: Suppose A has enough injectives.

Then

$$\tau: K^+(\mathcal{I}) \rightarrow D^+(A)$$

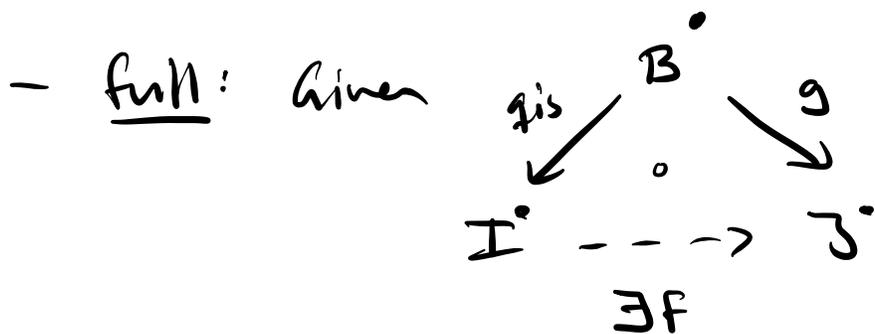
is an equiv. (of triangulated categories)

pf: Essentially surjective is Prop 2.35. (b/c enough inj.)

Fully faithful: Given $f: I \rightarrow J$ in $K^+(\mathcal{I})$.

- faithful: If $f = 0$ in $D^+(A)$, then $f \sim 0$, that is,

$$\begin{array}{ccc} & B^\bullet & \\ q \text{ is } \swarrow & \circ & \searrow \\ I^\bullet & \xrightarrow{f} & J^\bullet \end{array} \Rightarrow f \sim 0$$

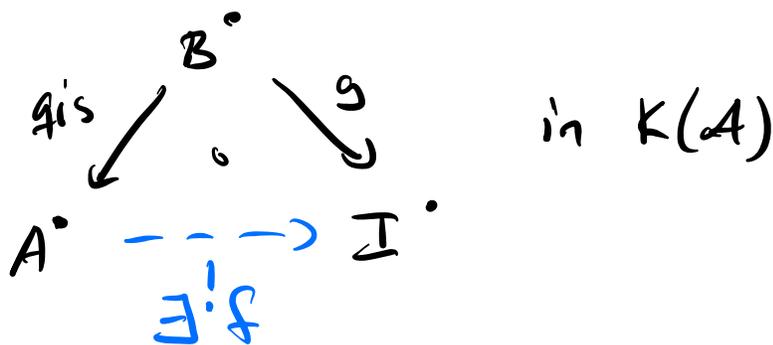


Both follows from:

Lem H 239 $A^\circ, I^\circ \in \text{Kom}^+(\mathcal{A})$ s.th. I° acy. $\forall i$. Then

$$\boxed{\text{Hom}_{K(\mathcal{A})}(A^\circ, I^\circ) \rightarrow \text{Hom}_{D(\mathcal{A})}(A^\circ, I^\circ)} \text{ bijective}$$

pf: Need to show



Equivalently:

Lemma H 2.38: $A^\circ, B^\circ \in K^+(\mathcal{A})$, $I^\circ \in K(\mathcal{A})$, $B^\circ \xrightarrow[s]{q_{is}} A^\circ$

Then:

$$\boxed{\text{Hom}_{K(\mathcal{A})}(A^\circ, I^\circ) \xrightarrow{\circ s} \text{Hom}_{K(\mathcal{A})}(B^\circ, I^\circ)}$$

is bijective.

$$\text{pf: } B^\circ \xrightarrow[q_{is}]{s} A^\circ \xrightarrow{\tilde{\tau}} C(s) \xrightarrow{\pi} B^\circ[1]$$

acyclic

\leadsto LES

$$\hookrightarrow \text{Hom}(C(s), I^\bullet) \rightarrow \text{Hom}(A^\bullet, I^\bullet) \rightarrow \text{Hom}(B^\bullet, I^\bullet)$$

$$\hookrightarrow \text{Hom}(C(s)[-1], I^\bullet) \rightarrow \text{Hom}(A^\bullet[-1], I^\bullet) \rightarrow \text{Hom}(B^\bullet[-1], I^\bullet)$$

so enough to prove

$$\boxed{\text{Hom}(C^\bullet, I^\bullet) = 0 \text{ if } C^\bullet \text{ acyclic}}$$

($\Leftrightarrow I^\bullet$ is K -injective)

Let $g: C^\bullet \rightarrow I^\bullet$. We will show that $g = 0$ in $K(A)$ that is $g = dh + hd$ for a homotopy h . By induction \exists

$$\begin{array}{ccccccc} \longrightarrow & C^{i-1} & \xrightarrow{d^{i-1}} & C^i & \xrightarrow{d^i} & C^{i+1} & \longrightarrow \\ & \searrow g^{i-1} & \downarrow & \searrow h^i & \downarrow g^i & \downarrow g^{i+1} & \\ & I^{i-1} & \xrightarrow{d^{i-1}} & I^i & \xrightarrow{d^i} & I^{i+1} & \longrightarrow \end{array}$$

s.th. $g^j = h^{j+1} d^j + d^{j-1} h^j$
 $\forall j < i$

(base case trivial because C^\bullet left bounded)

Consider $g^i - d^{i-1} \circ h^i: C^i \rightarrow I^i$.

This is zero on $\text{im}(C^{i-1} \xrightarrow{d^{i-1}} C^i)$ b/c

$$g^i \circ d^{i-1} - d^{i-1} \circ h^i \circ d^{i-1} = g^i \circ d^{i-1} - d^{i-1} \circ (g^{i-1} - d^{i-2} \circ h^{i-1}) = 0$$

$$\text{so } g^i - d^{i-1} \circ h^i: C^i / C^{i-1} \rightarrow I^i \text{ inj}$$

$$\begin{array}{ccc} \text{inj} \downarrow & & \nearrow h^{i+1} \\ C^{i+1} & & E \end{array}$$

and by construction $g^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$. □

Consequence: Suppose $A^\circ \xrightarrow{f} I^\circ$ inj. resolution
 $\searrow g \quad \downarrow \varphi$
 J°

Then $\text{Hom}_{K(A)}(I^\circ, J^\circ) \xrightarrow{\circ f} \text{Hom}_{K(A)}(A^\circ, J^\circ)$ bijective.

So $\exists! I^\circ \xrightarrow{\varphi} J^\circ$ up to homotopy s.t. $\varphi \circ f = g$.

§4 Derived functors

$F: \mathcal{A} \rightarrow \mathcal{B}$ additive

$$\rightsquigarrow \text{Kom}(\mathcal{A}) \xrightarrow{F} \text{Kom}(\mathcal{B})$$

$$(A^i, d^i) \longmapsto (F(A)^i, F(d^i))$$

$$\rightsquigarrow K(\mathcal{A}) \xrightarrow{K(F)} K(\mathcal{B}) \quad \underline{\text{triangulated}} \quad (= \text{exact})$$

b) (1) $F(A^\bullet[1]) = F(A^\bullet)[1]$

$$(2) F \left(\begin{array}{ccccc} A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{g} & C(F) \xrightarrow{\pi} A^\bullet[1] \\ & & & \parallel & \\ & & & B^\bullet \oplus A^\bullet[1] & \end{array} \right)$$

$$= \left(\begin{array}{cccc} F(A^\bullet) & \rightarrow & F(B^\bullet) & \rightarrow & F(A^\bullet[1]) \\ & & \parallel & & \parallel \\ & & F(B^\bullet) \oplus F(A^\bullet[1]) & & F(A^\bullet[1]) \end{array} \right)$$

i.e., $F(\text{triangle}) = \text{triangle}$.

Lemma H 2.4 Suppose $F: K(\mathcal{A}) \rightarrow K(\mathcal{B})$ exact. Then

$$\begin{array}{ccc} K(\mathcal{A}) & \xrightarrow{F} & K(\mathcal{B}) \\ \downarrow & \circ & \downarrow \\ D(\mathcal{A}) & \xrightarrow{F} & D(\mathcal{B}) \end{array} \quad (*)$$

iff

(1) $F(\text{qiso})$ is qiso \iff (2) $F(\text{acyclic})$ is acyclic.

pf: Use that any $G: D(\mathcal{A}) \rightarrow D(\mathcal{B})$ necessarily takes qiso to qiso. \square