Derived categories

Lecture 6

Feb 26, 2026

§1 Bounded versions

Def: $\mathrm{Kom}^-(\mathcal{A}) = \{ A^\bullet : A^i = 0 \ \forall i \gg 0 \}$ "bounded below"

$\mathrm{Kom}^+(\mathcal{A}) = \{ \quad \forall i \ll 0 \}$ "bounded above"

$\mathrm{Kom}^b(\mathcal{A}) = \{ \quad \forall |i| \gg 0 \}$ "bounded"

$= \mathrm{Kom}^- \cap \mathrm{Kom}^+$

$$K^*(\mathcal{A}) = \mathrm{Kom}^*(\mathcal{A}) / \sim \qquad * \in \{+, -, b\}$$

$$D^*(\mathcal{A}) = K^*(\mathcal{A}) / \text{qiso}$$

Prop H 2.30: $D^*(\mathcal{A}) \xrightarrow{\phi} D(\mathcal{A})$ fully faithful w/ essential image

$\{ A^\bullet : H^i(A^\bullet) = 0 \ \forall i \gg 0 \}$ "coh. bounded below"

$\forall i \ll 0$ "coh bounded above"

$\forall |i| \gg 0$ "coh bounded"

pf: Suppose $H^i(A^\bullet) = 0 \ \forall i > N$. Then $\tau^{\leq N} A^\bullet \to A^\bullet$ qiso so $A^\bullet \cong \phi(\tau^{\leq N} A^\bullet)$ in ess image of ϕ.

Similarly if $H^i(A^\bullet) = 0 \ \forall i < N$, then $A^\bullet \to \tau^{\geq N} A^\bullet$ qiso. etc

Similarly if $H^i(A^\bullet) = 0 \ \forall |i| > N$, then

$$\begin{tikzcd} & \tau^{\leq -N} A^\bullet \arrow[dl, "\text{qiso}"'] \arrow[dr, "\text{qiso}"] & \\ A^\bullet & & \tau^{\geq N} \tau^{\leq -N} A^\bullet \end{tikzcd}$$

etc

For fully faithful: if $A^i = B^i = 0 \ \forall i > N$

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ \phi(A^\bullet) & & \phi(B^\bullet) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & \tau^{\leq N} C^\bullet \arrow[d, "qis"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & C^\bullet \arrow[dl] \arrow[dr] & \\ \phi(A^\bullet) & & \phi(B^\bullet) \end{tikzcd}$$

etc

roof in $D^-(\mathcal{A})$

$\square$

<u>Rmk:</u> Not true in $K(\mathcal{A})$.

<u>Rmk:</u> Same arg. as $\mathcal{A} \longrightarrow D(\mathcal{A})$ fully faithful prev. lecture.

§2 Injective resolutions (of bounded below complexes)

Recall: $\mathcal{A}$ has enough injectives if $\forall A \in \mathcal{A}$ $\exists A \xhookrightarrow{\text{inj}} I$, I inj

— " — projectives — " — $\exists P \xrightarrow{\text{surj}} A$, P proj

Recall: If $\mathcal{A}$ has enough inj., $A \in \mathcal{A}$, $\exists$ inj. resolution $A \to I^\bullet$

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, equal] & I^0 \arrow[r, two heads] \arrow[d, equal] & Q^1 \arrow[d, hook] & & \\
A \arrow[r, hook] \arrow[d, equal] & I^0 \arrow[r] \arrow[d, equal] & I^1 \arrow[r, two heads] \arrow[d, equal] & Q^2 \arrow[d, hook] & \\
A \arrow[r, hook] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r, two heads] & Q^3
\end{tikzcd}$$
etc

GM III.5.25

Prop H 2.35: Suppose $\mathcal{A}$ has enough inj. Then $\forall A^\bullet \in Kom^+(\mathcal{A})$ $\exists I^\bullet \in Kom^+(\mathcal{A})$ w/ I^i inj $\forall i$ and $A^\bullet \to I^\bullet$ qiso.

pf: WLOG $A^\bullet = 0 \to 0 \to A^0 \to A^1 \to A^2 \to \cdots$

Step 0:

$$\begin{tikzcd}
0 \arrow[r] & A^0 \arrow[r, "d^0"] \arrow[d, hook, "f^0"', "\text{inj}"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$
I^0 inj

Then $H^0(A^\bullet) \xrightarrow{H^0(f)} H^0(I^\bullet)$ is injective.

$$\begin{tikzcd}
H^0(A^\bullet) \arrow[d, equal] \arrow[rr, "H^0(f)"] & & H^0(I^\bullet) \arrow[d, equal] \\
\ker d^0 \subset A^0 \arrow[rr, hook, "f^0"] & & I^0
\end{tikzcd}$$

<u>Step 1</u>:

$$\begin{tikzcd} A^0 \arrow[r, "d^0"] \arrow[d, "f^0"'] \arrow[dr, phantom, "0"] & A^1 \arrow[d] \\ I^0 \arrow[r] & I^0 \amalg_{A^0} A^1 \arrow[r, hook, "\mathrm{inj}"] & I_1 \end{tikzcd}$$

inj

$$\cong$$

$$\operatorname{coker}\left(A^0 \xrightarrow{\iota_2 \circ d^0 - \iota_1 \circ f^0} I^0 \oplus A^1\right) = I^0 \oplus A^1 \Big/ \left\langle \left(-f^0(a^0), d^0(a^0)\right) : a^0 \in A^0 \right\rangle$$

gives

$$\begin{tikzcd} A^0 \arrow[r, "d^0"] \arrow[d, "f^0"'] \arrow[dr, phantom, "0"] & A^1 \arrow[r] \arrow[d, "f^1"] & A^2 \arrow[r] \arrow[d] & A^3 \arrow[r] \arrow[d] & \cdots \\ I^0 \arrow[r, "d_I^0"] & I^1 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

<u>Claim</u>: $H^0(f)$ isomorphism
$H^1(f)$ injective.

<u>pf</u>: •

$$\begin{tikzcd} H^0(A^\bullet) \arrow[r, "H^0(f^\bullet)"] \arrow[d, phantom, "\cong"] & H^0(I^\bullet) \arrow[d, phantom, "\cong"] \\ \ker d^0 \arrow[r, "\tilde{f}^0"] \arrow[dr, "f^0"'] & \ker d_I^0 \arrow[d, phantom, "\cap"] \\ & I^0 \end{tikzcd}$$

- $\tilde{f}^0$ inj. b/c f^0 inj

- $\tilde{f}^0$ surj b/c

$i^0 \in \ker d_I^0 \iff (i^0, 0) = (0, 0)$ in $I^0 \amalg_{A^0} A^1$

(Here I use <u>elements</u>) $\iff \exists a^0 \in A^0$ s.th. $f^0(a^0) = i^0$, $d^0(a^0) = 0$

$\iff i^0 \in f^0(\ker d^0)$.

- $$\begin{tikzcd} H^1(A^\bullet) \arrow[r, "H^1(f^\bullet)"] \arrow[d, phantom, "\shortparallel"] & H^1(I^\bullet) \arrow[d, phantom, "\shortparallel"] \\ \ker d^1/\operatorname{im} d^0 \arrow[r] \arrow[d, phantom, "\cap"] & I^1/d_I^0(I^0) \\ \overline{a^1} \in A^1/d^0(A^0) \arrow[ur, "\overline{f^1}"'] & \end{tikzcd}$$

(using elements)

$\overline{f^1}$ <u>injective</u> b/c: $f^1(a^1) = d_I^0(i^0)$ for some $i^0 \in I^0$

$\iff (0, a^1) \sim (i^0, 0)$ —"—

$\iff \exists a^0 \in A^0 : a^1 = d^0(a^0),\ i^0 = f^0(a^0)$

$\iff a^1 \in \operatorname{im} d^0$ ■

<u>Step $i+1$</u> Suppose

$$\begin{tikzcd} 0 \arrow[r] & A^0 \arrow[r] \arrow[d, "f^0"] & \cdots \arrow[r] & A^i \arrow[r] \arrow[d, "f^i"] & A^{i+1} \arrow[d] \\ 0 \arrow[r] & I^0 \arrow[r] & \cdots \arrow[r] & I^i \arrow[r] & 0 \end{tikzcd}$$

$H^j(f^\bullet)$ iso $\forall j < i$ and inj for $j = i$

$$\begin{tikzcd} A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "d^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[rrd, bend left] \arrow[rrrd, bend left, "f^{i+1}"] & & & \\ I^{i-1} \arrow[r, "d_I^{i-1}"] & I^i \arrow[r, two heads] & \operatorname{coker} d_I^{i-1} \arrow[rr] & & \operatorname{coker} d_I^{i-1} \coprod_{A^i} A^{i+1} \arrow[r, hook] & I^{i+1} \end{tikzcd}$$

$H^i(f^\bullet)$ still injective.

$H^i(f^\bullet)$ surjective by cons.

$H^{i+1}(f^\bullet)$ injective —"—

$d_I^i \circ d_I^{i-1} = 0$ by cons

so now in assumption for Step $i+2$.

Step ∞:

$$\begin{tikzcd} 0 \arrow[r] & A^0 \arrow[r] \arrow[d, "f^0"'] & A^1 \arrow[r] \arrow[d, "f^1"] & \cdots \\ 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \end{tikzcd}$$

$A^\bullet \longrightarrow I^\bullet$ qiso.

Cor H 2.36: If $A^\bullet \in D(\mathcal{A})$ coh. left bounded then $\exists A^\bullet \longrightarrow I^\bullet$ qiso.

pf: $A^\bullet \xrightarrow{\text{qis}} \tau^{\geq -N} A^\bullet \xrightarrow{\text{qis}} I^\bullet$ $\square$

Rmk: We only used that $\forall A \in \mathcal{A}$, $\exists A \hookrightarrow I$, $I \in \mathcal{I} \subset \mathcal{A}$

We never used that $I \in \mathcal{I}$ are injective.

Same argument thus works for other classes $\mathcal{I} \subset \mathcal{A}$.

§3 $D^+(\mathcal{A}) = K^+(\mathcal{I})$

Def: $\mathcal{I} = \{I \in \mathcal{A} : I \text{ inj}\} \subset \mathcal{A}$ full additive subcategory.

Rmk: $\mathcal{I}$ additive, not abelian. $0 \in \mathcal{I}$ and $(I, J \in \mathcal{I} \Rightarrow I \oplus J \in \mathcal{I})$

Def: $K^*(\mathcal{I}) = \mathrm{Kom}^*(\mathcal{I})/\sim \qquad * \in \{-, +, b\}$

triangulated category: dist triangles are triangles iso to cone triangles

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A[1] \qquad \text{w/ } A, B \in \mathrm{Kom}(\mathcal{I})$$
$$\Rightarrow C(f) \in \mathrm{Kom}(\mathcal{I})$$

Prop H.2.40: Suppose $\mathcal{A}$ has enough injectives. Then

$$\boxed{\iota : K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})}$$

is an equiv. (of triangulated categories)

pf: Essentially surjective is Prop 2.35. (b/c enough inj.)

Fully faithful: Given $f : I^\bullet \to J^\bullet$ in $K^+(\mathcal{I})$.

– faithful: If $f = 0$ in $D^+(\mathcal{A})$, then $f \sim 0$, that is,

$$\begin{tikzcd} & B^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr, "0"] \arrow[d, phantom, "0"] & \\ I^\bullet \arrow[rr, "f"] & {} & J^\bullet \end{tikzcd} \qquad \Rightarrow f \sim 0$$

- <u>full</u>: Given

$$\begin{tikzcd} & B^\bullet \arrow[dl, "qis"'] \arrow[dr, "g"] & \\ I^\bullet \arrow[rr, dashed, "\exists f"'] & \circ & J^\bullet \end{tikzcd}$$

Both follows from:

<u>Lem H 2.39</u> $A^\bullet, I^\bullet \in \mathrm{Kom}^+(\mathcal{A})$ s.th. I^i inj. $\forall i$. Then

$$\boxed{\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet)}$$ bijective

<u>pf</u>: Need to show

$$\begin{tikzcd} & B^\bullet \arrow[dl, "qis"'] \arrow[dr, "g"] & \\ A^\bullet \arrow[rr, dashed, "\exists ! f"'] & \circ & I^\bullet \end{tikzcd}$$ in $K(\mathcal{A})$

Equivalently:

<u>Lemma H 2.38</u>: $A^\bullet, B^\bullet \in K^+(\mathcal{A})$, $I^\bullet \in K(\mathcal{I})$, $B^\bullet \xrightarrow[s]{qis} A^\bullet$

Then:

$$\boxed{\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) \xrightarrow{\circ s} \mathrm{Hom}_{K(\mathcal{A})}(B^\bullet, I^\bullet)}$$

is bijective.

<u>pf</u>: $$\underset{qis}{B^\bullet \xrightarrow{s} A^\bullet} \xrightarrow{\tau} \underset{acyclic}{C(s)} \xrightarrow{\pi} B^\bullet[1]$$

$\rightsquigarrow$ LES

$\hookrightarrow \operatorname{Hom}(C(s), I^\bullet) \to \operatorname{Hom}(A^\bullet, I^\bullet) \to \operatorname{Hom}(B^\bullet, I^\bullet) \rangle$

$\hookrightarrow \operatorname{Hom}(C(s)[-1], I^\bullet) \to \operatorname{Hom}(A^\bullet[-1], I^\bullet) \to \operatorname{Hom}(B^\bullet[-1], I^\bullet) \rangle$

So enough to prove

$$\boxed{\operatorname{Hom}(C^\bullet, I^\bullet) = 0 \text{ if } C^\bullet \text{ acyclic}}$$

($\Leftarrow$ $I^\bullet$ is K-injective)

Let $g: C^\bullet \to I^\bullet$. We will show that $g = 0$ in $K(A)$, that is $g = dh + hd$ for a homotopy h. By induction $\exists$

$$\begin{tikzcd}
{} \arrow[r] & C^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "g^{i-1}"] \arrow[dl] & C^i \arrow[r, "d^i"] \arrow[d, "g^i"] \arrow[dl, "h^i"'] & C^{i+1} \arrow[r] \arrow[d, "g^{i+1}"] \arrow[dl, dashed] & {} \\
{} \arrow[r] & I^{i-1} \arrow[r, "d^{i-1}"] & I^i \arrow[r, "d^i"] & I^{i+1} \arrow[r] & {}
\end{tikzcd}$$

s.th. $g^j = h^{j+1}d^j + d^{j-1}h^j$ $\forall j < i$

(base case trivial because $C^\bullet$ left bounded)

Consider $g^i - d^{i-1} \circ h^i : C^i \to I^i$.

This is zero on $\operatorname{im}(C^{i-1} \xrightarrow{d^{i-1}} C^i)$ b/c

$$g^i \circ d^{i-1} - d^{i-1} \circ h^i \circ d^{i-1} = g^i \circ d^{i-1} - d^{i-1} \circ (g^{i-1} - d^{i-2} \circ h^{i-1}) = 0$$

so

$$\begin{tikzcd}
C^i / C^{i-1} \arrow[r, "g^i - d^{i-1} \circ h^i"] \arrow[d, "\text{inj}"'] & I^i \text{ inj} \\
C^{i+1} \arrow[ur, dashed, "h^{i+1}\ \exists"']
\end{tikzcd}$$

and by construction $g^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$. $\square$

Consequence: Suppose $\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[dr, "g"'] & I^\bullet \arrow[d, dashed, "\varphi"] \\ & J^\bullet \end{tikzcd}$ inj. resolutions

Then $\mathrm{Hom}_{K(A)}(I^\bullet, J^\bullet) \xrightarrow{\circ f} \mathrm{Hom}_{K(A)}(A^\bullet, J^\bullet)$ bijection.

$$\underset{g}{\Downarrow}$$

So $\exists ! \, I^\bullet \xrightarrow{\varphi} J^\bullet$ up to homotopy s.th. $\varphi \circ f = g$.

§4 Derived functors

$F : \mathcal{A} \longrightarrow \mathcal{B}$ additive

$\rightsquigarrow \quad \mathrm{Kom}(\mathcal{A}) \xrightarrow{F} \mathrm{Kom}(\mathcal{B})$

$\qquad (A^i, d^i) \longmapsto (F(A)^i, F(d^i))$

$\rightsquigarrow \quad K(\mathcal{A}) \xrightarrow{K(F)} K(\mathcal{B})$ <u>triangulated</u> (= exact)

b/c (1) $F(A^\bullet[1]) = F(A^\bullet)[1]$

(2) $F\left(A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} \underset{\substack{\| \\ B^\bullet \oplus A^\bullet[1]}}{C(f)} \xrightarrow{\pi} A^\bullet[1] \right)$

$= \left(F(A^\bullet) \longrightarrow F(B^\bullet) \longrightarrow \underset{\substack{\| \\ F(B^\bullet) \oplus F(A^\bullet)[1]}}{F(B^\bullet \oplus A^\bullet[1])} \longrightarrow \underset{\substack{\| \\ F(A^\bullet)[1]}}{F(A^\bullet[1])} \right)$

i.e., F(triangle) = triangle.

<u>Lemma H 2.4</u> Suppose $F : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ exact. Then

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\circ"] & K(\mathcal{B}) \arrow[d] \\ D(\mathcal{A}) \arrow[r, dashed, "\exists"] & D(\mathcal{B}) \end{tikzcd} \qquad (*)$$

iff

(1) F(qiso) is qiso $\Longleftrightarrow$ (2) F(acyclic) is acyclic.

pf: Use that any $G : D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ necessarily takes qiso to qiso □