Derived categories

Lecture 7

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§1 Derived functors: exact case (recollection from last time)

$F : \mathcal{A} \longrightarrow \mathcal{B}$ additive $\rightsquigarrow$ $\mathrm{Kom}(\mathcal{A}) \xrightarrow{F} \mathrm{Kom}(\mathcal{B})$

$(A^i, d^i) \longmapsto (F(A)^i, F(d^i))$

$\rightsquigarrow$ $K(\mathcal{A}) \xrightarrow{K(F)} K(\mathcal{B})$

b/c $\qquad$ triangulated (= exact)

(1) $F(A^\bullet[1]) = F(A^\bullet)[1]$

(2) $F\left(A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} \underset{\underset{B^\bullet \oplus A^\bullet[1]}{\parallel}}{C(f)} \xrightarrow{\pi} A^\bullet[1]\right)$

$$= \Big(F(A^\bullet) \longrightarrow F(B^\bullet) \longrightarrow \underset{\underset{F(B^\bullet)\oplus F(A^\bullet)[1]}{\parallel}}{F(B^\bullet \oplus A^\bullet[1])} \longrightarrow \underset{\underset{F(A^\bullet)[1]}{\parallel}}{F(A^\bullet[1])} \Big)$$

i.e., F(triangle) = triangle.

TFAE: (H, Lemma 2.4)

(a) F exact

(b) $KF(\text{acyclic}) = \text{acyclic}$

(c) $KF(\text{qis}) = \text{qis}$

(d)
$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "KF"] \arrow[d] \arrow[dr, phantom, "\circ"] & K(\mathcal{B}) \arrow[d] \\ D(\mathcal{A}) \arrow[r, dashed, "\exists"] & D(\mathcal{B}) \end{tikzcd} \qquad (*)$$

proof:

(a) $\Leftrightarrow$ (b) by def

(b) $\Leftrightarrow$ (c) cone of qis

(c) $\Leftrightarrow$ (d) any $D(A) \to D(B)$ takes qis to qis.

__Non-ex:__ Suppose F left exact, then:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & C \arrow[r] & 0 & \text{SES} \\
& A \arrow[r] & B \arrow[r] & C(f) \arrow[r] \arrow[u, "\text{qiso}"'] & A[1] &
\end{tikzcd}$$

$$\begin{tikzcd}
\overset{F}{\longmapsto} & 0 \arrow[r] & F(A) \arrow[r, "f"] & F(B) \arrow[r] & F(C) \\
& & & & F(B)/F(A) \arrow[u, "\leftsquigarrow \text{ qiso exactly when iso}"'] \\
& & F(A) \arrow[r, "F(f)"] & F(B) \arrow[r] & C(F(f)) \arrow[u, "\text{qiso}"']
\end{tikzcd}$$

For $F : \mathcal{A} \to \mathcal{B}$ not exact cannot get (*) commutative

Want best possible replacement. (next section)

§2 Right-derived functors

Prop H 2.47⁺ Suppose $F : K^+(\mathcal{A}) \to \mathcal{D}$ exact. If $\mathcal{A}$ has enough injectives then $\exists$ right-derived functor:

$$\exists \quad \begin{tikzcd} K^+(\mathcal{A}) \arrow[d, "Q"'] \arrow[dr, "F"] \arrow[d, phantom, shift left=4ex, "\eta \swArrow"] & \\ D^+(\mathcal{A}) \arrow[r, "RF"'] & \mathcal{D} \end{tikzcd}$$

that is,

(1) an exact functor $RF : D^+(\mathcal{A}) \to \mathcal{D}$

(2) a natural transformation $\eta : F \Rightarrow RF \circ Q$

Moreover (RF, η) has the following universal property:

$$\forall \quad \begin{tikzcd} K^+(\mathcal{A}) \arrow[d, "Q"'] \arrow[dr, "F"] \arrow[d, phantom, shift left=4ex, "\varphi \swArrow"] & \\ D^+(\mathcal{A}) \arrow[r, "G"'] & \mathcal{D} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} K^+(\mathcal{A}) \arrow[d] \arrow[dr] \arrow[d, phantom, shift left=4ex, "\eta \swArrow"] & \\ D^+(\mathcal{A}) \arrow[r, bend left, "RF"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \exists! \psi"] & \mathcal{D} \end{tikzcd}$$

that is: $\exists! \psi : RF \Rightarrow G$

s.th. $\psi \circ \eta = \varphi$.

Rmk: This means that RF is the left (!) Kan extension of F along Q.

[Rmk: That $\mathcal{A}$ has enough injectives can be replaced w/ $K^+(\mathcal{A})$ having enough F-acyclic. (adapted class, see later)]

Construction of RF (and η)   [now starting w/ $K^+(\mathcal{A}) \xrightarrow{F} D(\mathcal{B})$ rather than $\mathcal{A} \xrightarrow{F} \mathcal{B}$]

$$\begin{tikzcd}
K^+(\mathcal{I}) \arrow[r, hook, "\text{f.f.}"] \arrow[dr, "\iota"', "\text{equiv}"] & K^+(\mathcal{A}) \arrow[d, "Q"] \arrow[dr, "F"] & \\
 & D^+(\mathcal{A}) \arrow[r, "RF"'] & D
\end{tikzcd}$$

$$\boxed{RF := F \circ \iota^{-1}}$$

Fact (H 1.41) ι^{-1} is exact. $\Rightarrow$ RF exact.

Quasi-inverse ι^{-1} exists since ι equiv of categories. Comes w/ natural isomorphism $id_{D^+(\mathcal{A})} \xrightarrow{\eta'} \iota \circ \iota^{-1}$. Explicitly, if $A^\bullet \in D^+(\mathcal{A})$ then

$$\iota^{-1}(A^\bullet) = I^\bullet \quad \text{injective res}$$

and

$$\eta' : A^\bullet \xrightarrow{\cong} \iota(\iota^{-1}(A^\bullet)) \quad \text{in } D^+(\mathcal{A}) \quad \text{functorial in } A^\bullet$$

$$\Longleftrightarrow \quad \begin{tikzcd}[column sep=small] & C^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr, "\text{qis}"] & \\ A^\bullet & & \iota^{-1}(A^\bullet) \end{tikzcd} \quad \text{in } K^+(\mathcal{A}) \quad \text{— " —}$$

$$\overset{2.38}{\Longleftrightarrow} \quad A^\bullet \xrightarrow{\text{qis}} \iota^{-1}(A^\bullet) \quad \text{in } K^+(\mathcal{A}) \quad \text{— " —}$$

So ι^{-1} gives functorial choice of inj. resolutions for all $A^\bullet$.

Applying F gives $F(A^\bullet) \xrightarrow{\eta} F(\iota^{-1}(A^\bullet)) = RF(A^\bullet)$ funct. in $A^\bullet$.

§3 Standard setting:

$F : \mathcal{A} \longrightarrow \mathcal{B}$ left-exact, $\mathcal{A}$ has enough injectives

$\rightsquigarrow$

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[r, "KF"] \arrow[d, "Q_{\mathcal{A}}"'] \arrow[dr, phantom, "\swarrow\eta"] & K^+(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ D^+(\mathcal{A}) \arrow[r, dashed, "RF"'] & D^+(\mathcal{B}) \end{tikzcd}$$

If $A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$ inj. resolution, then

$$RF(A[0]) = F(I^0) \longrightarrow F(I^1) \longrightarrow \cdots$$

<u>Def</u>: Right-derived functors of F are $R^iF : \mathcal{A} \longrightarrow \mathcal{B}$ $(i \in \mathbb{Z})$

$$(R^iF)(A) := H^i(RF(A[0])) = H^i(F(I^\bullet))$$

<u>Rmk</u>:

- $(R^iF)(A) = 0 \quad \forall i < 0$
- $(R^0F)(A) = F(A)$

b/c $0 \to A \to I^0 \xrightarrow{d^0} I^1 \to \cdots \quad \xRightarrow{F \text{ left-exact}} \quad \overbrace{0 \to F(A) \to F(I^0) \xrightarrow{F(d^0)} F(I^1)}^{\text{exact}} \to \cdots$

$$F(A) = (R^0F)(A) := \ker(F(d^0))$$

<u>Upshot</u>: RF <u>measures non-exactness</u> of F:

<u>Cor. 2.50</u> F left-exact, $0 \to A \to B \to C \to 0$ SES in $\mathcal{A}$ gives LES

$$0 \to FA \to FB \to FC \to (R^1F)(A) \to (R^1F)(B) \to (R^1F)(C) \to \cdots$$

in $\mathcal{B}$.

<u>proof</u>: $A[0] \to B[0] \to C[0] \to A[1]$ $\triangle$ in $D^+(\mathcal{A})$

$\Rightarrow$ $RF(A[0]) \to RF(B[0]) \to RF(C[0]) \to RF(A[0])[1]$ $\triangle$ in $D^+(\mathcal{B})$

$\Rightarrow$ LES after taking $H^\bullet(-)$. $\square$

<u>Def</u>: $A \in \mathcal{A}$ **F-acyclic** if $R^iF(A) = 0$ $\forall i > 0$.

<u>Ex</u>: injective $\Rightarrow$ F-acyclic (for right-derived functors)

Similar story for right-exact, enough projectives, left-derived functors.

§4 Examples

$0 \to A \to B \to C \to 0$ SES in $\mathcal{A}$

$M \in \mathcal{A}$ fixed object.

① $\boxed{\mathrm{Hom}(M,-)} : \mathcal{A} \to \mathrm{Ab}$ left-exact.

$\rightsquigarrow \boxed{R\mathrm{Hom}(M,-)} : D^+(\mathcal{A}) \to D^+(\mathrm{Ab})$ right-derived functor

$\rightsquigarrow \boxed{\mathrm{Ext}^i(M,-)} := R^i\mathrm{Hom}(M,-)$

$$0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C) \to \mathrm{Ext}^1(M,A) \to \cdots$$

e.g. $\mathcal{A} = \mathrm{Mod}_R$

② $\mathcal{A}$ also monoidal, $\boxed{M \otimes -} : \mathcal{A} \to \mathcal{A}$ right-exact

$\rightsquigarrow \boxed{M \overset{\mathbb{L}}{\otimes} -} : D^-(\mathcal{A}) \to D^-(\mathcal{A})$ left-derived functor

$\rightsquigarrow \boxed{\mathrm{Tor}_i(M,-)} := H^{-i}(M \overset{\mathbb{L}}{\otimes} -)$ (if $\mathcal{A}$ has enough proj)

$$\cdots \to \mathrm{Tor}_1(M,A) \to M \otimes A \to M \otimes B \to M \otimes C \to 0$$

③ $\mathcal{A}$ closed monoidal $\Rightarrow \exists \boxed{\mathcal{H}om(M,-)} : \mathcal{A} \to \mathcal{A}$ left-exact

internal Hom

$\rightsquigarrow \boxed{R\mathcal{H}om(M,-)} : D^+(\mathcal{A}) \to D^+(\mathcal{A})$

$\boxed{\mathcal{E}xt^i(M,-)} := R^i\mathcal{H}om(M,-)$

Rmk: $\mathcal{H}om(M,-)$ right adjoint to $M \otimes -$ (K-inj, K-flat, ...)

$R\mathcal{H}om(M,-)$ —— " —— to $M \overset{\mathbb{L}}{\otimes} -$ if extended to $D(\mathcal{A})$

§5 More on Hom

Prop H 2.56 $\mathrm{Ext}^i(A,B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, B[i])$

proof: $B \xrightarrow{\text{qis}} (I^0 \to I^1 \to I^2 \to \cdots)$

$$\mathrm{LHS} = H^i(\cdots \to \mathrm{Hom}(A, I^{i-1}) \to \mathrm{Hom}(A, I^i) \to \mathrm{Hom}(A, I^{i+1}) \to \cdots)$$

$$h \longmapsto d^{i-1} \circ h \qquad f^i \longmapsto d^i \circ f^i = 0$$

$$\mathrm{RHS} = \mathrm{Hom}_{K^+(\mathcal{A})}(A, I^\bullet[i]) = \left\{ \begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f^i"] \arrow[dl, "h"'] & 0 \arrow[d] \\ I^{i-1} \arrow[r, "d^{i-1}"'] & I^i \arrow[r, "d^i"'] & I^{i+1} \end{tikzcd} \right\} \Big/ \text{hpts } h$$

$$d^i \circ f^i = 0$$

$$\mathrm{LHS} = \text{cycles}/\text{boundaries} = \text{complex maps}/\text{hptys} = \mathrm{RHS} \qquad \square$$

Consequence: $A \xrightarrow{g} B[i]$, $B \xrightarrow{f} C[j] \overset{\text{comp}}{\rightsquigarrow} A \xrightarrow{g} B[i] \xrightarrow{f[i]} C[i+j]$

$$\Rightarrow \mathrm{Ext}^i(A,B) \times \mathrm{Ext}^j(B,C) \longrightarrow \mathrm{Ext}^{i+j}(A,B).$$

"complex Hom"

Def: $A^\bullet, B^\bullet \in \mathrm{Kom}(\mathcal{A}) \rightsquigarrow \boxed{\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)} \in \mathrm{Kom}(\mathrm{Ab})$

$$\mathrm{Hom}^i(A^\bullet, B^\bullet) := \bigoplus_k \mathrm{Hom}(A^k, B^{k+i})$$

$$d(f) = d_B \circ f - (-1)^i f \circ d_A$$

Rmk: $H^0(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)) = \text{complex maps}/\text{homotopies} = \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet)$

Right-derived: 2^{nd} argument (if enough injectives)

$$\boxed{\mathrm{Hom}^\bullet(A^\bullet, -)} : K(\mathcal{A}) \longrightarrow K(Ab) \quad \text{exact}$$

$$\rightsquigarrow \boxed{R\mathrm{Hom}^\bullet(A^\bullet, -)} : D^+(\mathcal{A}) \longrightarrow D(Ab)$$

calculated as $R\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) := \mathrm{Hom}^\bullet(A^\bullet, I^\bullet)$
for $B^\bullet \xrightarrow{qis} I^\bullet$

Rmk: Only depends on $A^\bullet$ up to qis

$$\rightsquigarrow R\mathrm{Hom}^\bullet(-,-) : D(\mathcal{A})^{op} \times D^+(\mathcal{A}) \longrightarrow D(Ab)$$

Right-derived: 1^{st} argument

$$\boxed{\mathrm{Hom}^\bullet(-, B^\bullet)} : K(\mathcal{A})^{op} \longrightarrow K(Ab) \quad \text{exact}$$

projective resolution $P^\bullet \xrightarrow{qis} A^\bullet$ in $K(\mathcal{A})$
$\Leftrightarrow$ injective resolution $A^{-\bullet} \rightarrow P^{-\bullet}$ in $K(\mathcal{A}^{op}) = K(\mathcal{A})^{op}$

If $\exists$ enough projectives in $\mathcal{A}$ $\Rightarrow$ $\exists$ enough inj. in $\mathcal{A}^{op}$

$$\rightsquigarrow \boxed{R\mathrm{Hom}^\bullet(-, B^\bullet)} : \underset{\substack{\cong \\ D^-(\mathcal{A})^{op}}}{D^+(\mathcal{A}^{op})} \longrightarrow D(Ab)$$

Calculated as $R\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) = \mathrm{Hom}^\bullet(P^\bullet, B^\bullet)$

$$\rightsquigarrow R\mathrm{Hom}^\bullet(-,-) : D^-(\mathcal{A})^{op} \times D(\mathcal{A}) \longrightarrow D(Ab)$$

Fact: Agrees on $D^-(A)^{op} \times D^+(A)$ if both enough inj & proj.

Rmk: $R^0\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) := H^0(\mathrm{Hom}^\bullet_{K^+(A)}(A^\bullet, I^\bullet))$

$$= H^0(\mathrm{Hom}^\bullet_{D(A)}(A^\bullet, B^\bullet))$$

so $R\mathrm{Hom}^\bullet$ makes $D(A)$ enriched in $D(Ab)$.

(leads to $D(A)$ as ∞-category)

Also: $R^i\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) = \mathrm{Hom}_{D^+(A)}(A^\bullet, B^\bullet[i])$

Rmk: If $\mathcal{A}$ has an internal $\mathcal{H}om$, also get

$$\mathcal{H}om^\bullet(A^\bullet, B^\bullet) \in \mathrm{Kom}(\mathcal{A})$$
$$R\mathcal{H}om^\bullet(A^\bullet, B^\bullet) \in D(\mathcal{A})$$

§6 Adapted classes (when not enough inj/proj)

Recall: $K^+(\mathcal{I}) \xrightarrow{\text{equiv}} D^+(\mathcal{A})$ if enough inj.

<u>Def</u>: Let $F : \mathcal{A} \to \mathcal{B}$. A set of objects $\mathcal{R} \subset \mathcal{A}$ is **F-adapted** or **F-injective** if

(0) $I, J \in \mathcal{R} \Rightarrow I \oplus J \in \mathcal{R}$.

(1) $I^\bullet \in K^+(\mathcal{R})$ then $KF(I^\bullet)$ acyclic. "$\mathcal{R}$ F-acyclic"

(2) $\forall A \in \mathcal{A}$, $\exists I \in \mathcal{R}$ and $A \hookrightarrow I$. "enough F-injectives"

<u>Prop GM III 6.4</u>: If $\mathcal{R}$ F-adapted (or merely cond (0)+(2)), then

$$K^+(\mathcal{R})[\text{qis}^{-1}] \longrightarrow D^+(\mathcal{A}) \quad \text{equiv. of } \Delta \text{ categories.}$$

<u>proof sketch</u>: (2) $\Rightarrow$ ess. surj.

For fully faithful: $A^\bullet, B^\bullet \in K^+(\mathcal{R})$

$$\begin{tikzcd} & C^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd} \in D^+(\mathcal{A}) \quad \rightsquigarrow \quad \begin{tikzcd} A^\bullet \arrow[dr] & & B^\bullet \arrow[dl, "\text{qis}"] \\ & D^\bullet & \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd} A^\bullet \arrow[dr] \arrow[ddr, bend right] & & B^\bullet \arrow[dl, "\text{qis}"] \arrow[ddl, bend left, "\text{qis}"] \\ & D^\bullet \arrow[d, "\text{qis}"] & \\ & I^\bullet & \end{tikzcd} \in K^+(\mathcal{R}) \quad \rightsquigarrow \quad \begin{tikzcd} & J^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd} \in K^+(\mathcal{R})[\text{qis}^{-1}]$$

$\square$

Thm (GM, III.6.8) If a left-exact functor $F: \mathcal{A} \to \mathcal{B}$ admits an adapted class $\mathcal{R}$ then $RF: D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists. Moreover $RF = \overline{F} \circ \iota^{-1}$:

$$\begin{tikzcd}
K^+(\mathcal{R}) \arrow[r, hook] \arrow[d, "Q_{\mathcal{R}}"'] & K^+(\mathcal{A}) \arrow[r] \arrow[d, "Q_{\mathcal{A}}"] & K^+(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\
K^+(\mathcal{R})[qis^{-1}] \arrow[r, "\text{equiv}", "\iota"'] \arrow[rr, bend right, "\overline{F} \text{ b/c } \mathcal{R}\ F\text{-acyclic}"'] & D^+(\mathcal{A}) \arrow[r, dashed, "RF"] & D^+(\mathcal{B})
\end{tikzcd}$$

Ex: $F = M \otimes - : \mathcal{A} \to \mathcal{A}$, then flat objects are F-adapted

$\{\text{projective}\} \subset \{\text{flat}\}$ easier to have enough flats than projectives

$M \overset{\mathbb{L}}{\otimes} A$ can be calculated by taking a flat resolution $F^\bullet \to A$ instead of a proj resolution.

$\| $

$M \overset{\mathbb{L}}{\underset{A}{\otimes}} F^\bullet$

One can also talk about adapted for functors defined on $K^+(\mathcal{A})$ instead of on $\mathcal{A}$.

Def: $\mathcal{K} \subset K^+(\mathcal{A})$ (right-) adapted to $F: K^+(\mathcal{A}) \longrightarrow K(\mathcal{B})$ if

(a) $I^\bullet \in \mathcal{K}$ acyclic, then $F(I^\bullet)$ acyclic. "$\mathcal{K}$ is F-acyclic"

(b) $\forall A^\bullet \in K^+(\mathcal{A})$, $\exists$ qis $A^\bullet \longrightarrow I^\bullet$. "has enough $\mathcal{K}$-resolutions"

Thm (Yekutieli, arXiv, 8.3.3) If $\exists \mathcal{K}$ adapted to F, then RF exists.

Ex: Given $M^\bullet \in \mathrm{Kom}(\mathcal{A})$, have $M^\bullet \otimes - : \mathrm{Kom}(\mathcal{A}) \longrightarrow \mathrm{Kom}(\mathcal{A})$

$$K^-(\mathcal{A}) \longrightarrow K(\mathcal{A})$$

$K^-(\text{flats}) \subset K^-(\mathcal{A})$ adapted to $M^\bullet \otimes -$

$$\rightsquigarrow \quad M^\bullet \overset{\mathbb{L}}{\otimes} - : D^-(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

if enough flats.