Derived categories

Lecture 8

Mar 12, 2026

§1 Schemes and sheaves

R noetherian ring $\rightsquigarrow$ possible abelian categories:

$$\mathrm{Mod}_R^{f.g.} \subset \mathrm{Mod}_R$$

not enough inj — enough inj + proj

$X = \mathrm{Spec}\, R$ affine noetherian scheme:

$$\mathrm{Coh}(X) \cong \mathrm{Mod}_R^{f.g.} \subset \mathrm{QCoh}(X) \cong \mathrm{Mod}_R \subset \mathrm{Mod}_{\mathcal{O}_X} = \mathrm{Sh}_{\mathcal{O}_X}(X) \ni j_! \mathcal{O}_U$$

$$\mathcal{F} \longmapsto \Gamma(X, \mathcal{F})$$
$$\tilde{M} \longleftarrow M$$

Ex: $j : U \hookrightarrow X$ open

$j_! \mathcal{O}_U$ non-quasicoherent $\mathcal{O}_X$-module

$$(j_! \mathcal{F})(V) = \begin{cases} \mathcal{F}(V) & \text{if } V \subseteq U \\ 0 & \text{o/w} \end{cases}$$

X noeth scheme:

$$\mathrm{Coh}\, X \subset \mathrm{QCoh}\, X \subset \mathrm{Mod}_{\mathcal{O}_X}$$

$\mathrm{Coh}\, X$	$\mathrm{Mod}_{\mathcal{O}_X}$
not enough inj's	enough inj's
not enough flats	enough flats
not enough proj's	not enough proj's

$$D^b(X) := \boxed{D^b(\mathrm{Coh}\, X)} \subset D^b(\mathrm{QCoh}\, X) \subset D(\mathrm{Mod}_{\mathcal{O}_X})$$

main interest — enough inj etc

Notation:
$$H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F})$$
$$\mathcal{H}^i(\mathcal{F}^\bullet) = \ker d^i / \mathrm{im}\, d^{i-1} \quad (\text{prev. denoted } H^i)$$
(some sources use h^i)

§2 Quasi-coherent cohomology vs quasi-coherent

Def: $D_{qc}(\mathrm{Mod}_{\mathcal{O}_X}) = \{ \mathcal{F}^\bullet \in D(\mathrm{Mod}_{\mathcal{O}_X}) : \mathcal{H}^i(\mathcal{F}^\bullet) \in \mathrm{QCoh}(X) \}$

[H] Cor 3.4 X noetherian. Then $D^+(\mathrm{QCoh}\, X) \xrightarrow{\text{equiv}} D^+_{qc}(\mathrm{Mod}_{\mathcal{O}_X})$
(and sim. for D^b)

Follows from:

Prop [H, 3.3] X noetherian. $\mathcal{F} \in \mathrm{QCoh}(X)$. Then $\exists$ resolution:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \cdots$$

where $\mathcal{I}^i \in \mathrm{QCoh}(X)$ and injective as $\mathcal{O}_X$-modules
($\Rightarrow$ injective as qcoh. sheaves)

<u>proof</u>: uses X noeth:
- $\mathcal{I}$ injective $\Longleftrightarrow$ $\mathcal{I}_x$ inj $\forall x \in X$
- $\oplus$ inj is inj
- indecomposable injectives are injective hull of $\kappa(x)$.

(see Hartshorne, Residues and Duality II.7)

§3 Coherent cohomology vs coherent

Prop [H, 3.5]: X noeth. $D^b(X) := D^b(\mathrm{Coh}\, X) \xrightarrow{\text{equiv}} D^b_{\mathrm{coh}}(\mathrm{QCoh})$

Proof: **Ess surj:** Let $\mathcal{G} = (0 \to \mathcal{G}^n \to \mathcal{G}^{n+1} \to \cdots \to \mathcal{G}^m \to 0) \in \mathrm{RHS}$

Step m:

$$\begin{tikzcd}
\mathcal{G}^{m-1} \arrow[r] \arrow[d] & \overset{\text{qcoh}}{\mathcal{G}^m} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \underset{\text{coh}}{\mathcal{H}^m} \arrow[r] & 0
\end{tikzcd}$$

Fact ($\sim$[H, 3.6]) X noeth, $\mathcal{G} \in \mathrm{QCoh}(X)$. Then $\mathcal{G}$ union of its coherent subsheaves. (cf. R-module union of f.g. R-submodules)

Consequence: $\exists\, \mathcal{F}^m \subset \mathcal{G}^m$ coherent s.th. $\mathcal{F}^m \to \mathcal{G}^m \to \mathcal{H}^m$ surjective.

Let $\mathcal{F}^{m-1} = (d^{m-1})^{-1}(\mathcal{F}^m) \subseteq \mathcal{G}^{m-1}$ and $\mathcal{F}^i = \mathcal{G}^i$ $\forall i \leq m-2$.
Then have qiso $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{F}^{m-3} \arrow[r] \arrow[d, equal] & \mathcal{F}^{m-2} \arrow[r] \arrow[d, equal] & \mathcal{F}^{m-1} \arrow[r] \arrow[d, phantom, "\cap" rotate=-90] & \mathcal{F}^m \arrow[r] \arrow[d, phantom, "\cap" rotate=-90] & 0 \\
\cdots \arrow[r] & \mathcal{G}^{m-3} \arrow[r] & \mathcal{G}^{m-2} \arrow[r] & \mathcal{G}^{m-1} \arrow[r] & \mathcal{G}^m \arrow[r] & 0
\end{tikzcd}$$

Replace $\mathcal{G}^\bullet$ w/ $\mathcal{F}^\bullet$

Step $m-1$: $\quad \mathcal{G}^{m-2} \longrightarrow \mathcal{G}^{m-1} \longrightarrow \mathcal{G}^m$

①
$$\begin{tikzcd} \mathcal{G}^{m-1} \arrow[r, two heads] & \operatorname{im} d^{m-1} \subset \mathcal{G}^m \\ \exists\, \mathcal{F}_1^{m-1} \arrow[u, phantom, "\cup"] \arrow[ur, dashed, two heads] & \end{tikzcd}$$
$\mathcal{F}_1^{m-1}$ coh, $\operatorname{im} d^{m-1}$ coh, $\mathcal{G}^m$ coh

②
$$\begin{tikzcd} \ker d^{m-1} \arrow[r] & \mathcal{H}^{m-1}(\mathcal{G}^\bullet) \\ \exists\, \mathcal{F}_2^{m-1} \arrow[u, phantom, "\cup"] \arrow[ur, dashed, two heads] & \end{tikzcd}$$
$\mathcal{F}_2^{m-1}$ coh, $\mathcal{H}^{m-1}(\mathcal{G}^\bullet)$ coh

Let $\mathcal{F}^{m-1} = \mathcal{F}_1^{m-1} \cup \mathcal{F}_2^{m-1} \subset \mathcal{G}^{m-1}$

$\mathcal{F}^{m-2} = d^{-1}(\mathcal{F}^{m-1})$

$\mathcal{F}^i = \mathcal{G}^i \quad \forall i \neq m-1, m-2$

$\Rightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet$ qis and $\mathcal{F}^{m-1}, \mathcal{F}^m$ coherent.

⋮

Step n $\Rightarrow \mathcal{F}^\bullet \xrightarrow{\text{qis}} \mathcal{G}^\bullet$, $\mathcal{F}^\bullet \in D^b(\mathrm{Coh}\, X)$

Fully faithful: Use:

$$\begin{tikzcd} & \exists\, K^\bullet \in D^b(\mathrm{Coh}\, X) \arrow[d, "\text{qis}"'] & \\ & \mathcal{H}^\bullet \in D_{coh}(\mathrm{QCoh}\, X) \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ \mathcal{F}^\bullet & & \mathcal{G}^\bullet \end{tikzcd}$$

$\mathcal{F}^\bullet \in D^b(\mathrm{Coh}\, X)$, $\quad \mathcal{G}^\bullet \in D^b(\mathrm{Coh}\, X)$

$\square$

§4 Derived functors

Since $\mathrm{Mod}_{\mathcal{O}_X}$ has enough injectives and flats, can define derived functors. Have a new functor:

<u>Ex</u> (global sections) $\Gamma(X,-) : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Ab}$ left-exact

$\rightsquigarrow R\Gamma(X,-) : D^+(\mathcal{O}_X) \to D^+(\mathrm{Ab})$ triangulated

$\rightsquigarrow H^i(X,-) = R^i\Gamma(X,-) = H^i(R\Gamma(X,-)) : D^+(\mathcal{O}_X) \to \mathrm{Ab}$

$\uparrow$ sometimes denoted $\mathbb{H}^i$ when argument is complex.

	functor	derived functor	i^{th} der. functor
	$\Gamma(X,-)$	$R\Gamma(X,-)$	$H^i(X,-)$
already seen	$\mathrm{Hom}(\mathcal{F},-)$	$R\mathrm{Hom}(\mathcal{F},-)$	$\mathrm{Ext}^i(\mathcal{F},-)$
	$\mathcal{H}om(\mathcal{F},-)$	$R\mathcal{H}om(\mathcal{F},-)$	$\mathcal{E}xt^i(\mathcal{F},-)$
	$\mathcal{F}\otimes -$	$\mathcal{F}\overset{\mathbb{L}}{\otimes} -$	$\mathrm{Tor}_i(\mathcal{F},-)$
	$\mathrm{Hom}^\bullet(\mathcal{F}^\bullet,-)$	$R\mathrm{Hom}^\bullet(\mathcal{F}^\bullet,-)$	$\mathrm{Ext}^i(\mathcal{F}^\bullet,-)$
	$\mathcal{H}om^\bullet(\mathcal{F}^\bullet,-)$	$R\mathcal{H}om^\bullet(\mathcal{F}^\bullet,-)$	$\mathcal{E}xt^i(\mathcal{F}^\bullet,-)$
	$\mathcal{F}^\bullet\otimes -$	$\mathcal{F}^\bullet\overset{\mathbb{L}}{\otimes} -$	$\mathrm{Tor}_i(\mathcal{F}^\bullet,-)$

(where $- \in D^+(\mathcal{O}_X)$ or $D^-(\mathcal{O}_X)$ for right or left derived)

Since we have quasi-coherent injective resolutions, derived functors preserve qcoh whenever original functor does.

<u>Ex</u>
- $\mathcal{F}\otimes -$ preserves quasi-coherence
- $\Gamma(X,-)$ ——〃—— if X noeth (or qcqs)
- $\mathrm{Hom}(\mathcal{F},-)$ ——〃—— if $\mathcal{F}$ fin. pres (e.g. X noeth, $\mathcal{F}$ coh)
- $\mathcal{H}om(\mathcal{F},-)$ ——〃—— ——〃——

$\underline{\text{Ex}}$: (push-forward) $f : X \longrightarrow Y$ morphism of schemes

$$\rightsquigarrow \quad f_* : \mathrm{Mod}_{\mathcal{O}_X} \longrightarrow \mathrm{Mod}_{\mathcal{O}_Y}$$
$$\cup \qquad\qquad \cup$$
$$f_* : \mathrm{QCoh}\,X \dashrightarrow \mathrm{QCoh}(Y) \quad (\text{if } X, Y \text{ noeth.})$$

$$\rightsquigarrow \quad Rf_* : D^+(\mathcal{O}_X) \longrightarrow D^+(\mathcal{O}_Y) \qquad \rightsquigarrow \qquad R^i f_*$$
$$\cup \qquad\qquad \cup$$
$$D^+(\mathrm{QCoh}\,X) \longrightarrow D^+(\mathrm{QCoh}\,Y)$$

$\underline{\text{Q}}$: (composition) $\quad X \xrightarrow{f} Y \xrightarrow{g} Z$, then $g_* \circ f_* = (g \circ f)_*$

but is $\boxed{Rg_* \circ Rf_* \simeq R(g \circ f)_* \,?}$

RHS: $R(g \circ f)_* \mathcal{F}^\bullet = (g \circ f)_* I^\bullet = g_*(f_* I^\bullet)$ where $\mathcal{F}^\bullet \xrightarrow{\text{q.i.}} I^\bullet$

LHS: $Rg_* \circ Rf_* = Rg_*(f_* I^\bullet) = g_*(J^\bullet)$ where $f_* I^\bullet \xrightarrow{\text{q.i.}} J^\bullet$

$$\{\text{injective sheaves of } \mathcal{O}_Y\text{-mod}\} \subseteq \{\underline{g_*\text{-acyclic}}\ \mathcal{O}_Y\text{-mod } \mathcal{G}\}$$

$R^i g_* \mathcal{H} = 0 \ \forall i > 0$

g_*-adapted class

If $f_* I^i$ are g_*-acyclic, then we don't need to replace w/ an injective resolution to calculate $Rg_*(f_* I^\bullet)$.

$\underline{\text{Facts}}$ (Hartshorne, Alg Geom, III.2 + 8)

- injective $\Rightarrow$ flabby
- f_*(flabby) is flabby
- flabby $\Rightarrow$ g_*-acyclic $\qquad \Rightarrow$ answer is YES.

§5 Finiteness (for f_* and Γ)

Thm: $X \xrightarrow{f} Y$ b/w noetherian schemes, $\mathcal{F} \in QCoh\, X$
$R^i f_* \mathcal{F} = 0 \quad \forall i > \dim X.$

Thm: $X \longrightarrow Y$ proper b/w noetherian schemes.
Then $R^i f_* : Coh\, X \longrightarrow Coh\, Y$.
In particular $\boxed{Rf_* : D^b(X) \longrightarrow D^b(Y)}$

pf: Chow lemma to reduce to $\mathbb{P}^n$
explicit calc for $\mathcal{O}(d)$.

Special case $Y = \operatorname{Spec} k$. Then "$f_* = \Gamma$":

$$\begin{tikzcd} QCoh\, X \arrow[r, "f_*"] \arrow[ddr, "\Gamma(X,-)"'] & QCoh(\operatorname{Spec} k) \arrow[d, equal] \\ & Mod_k \arrow[d] \\ & Ab \end{tikzcd}$$

Cor: X proper scheme / $\operatorname{Spec} k$

(a) $\mathcal{F} \in Coh(X) \Rightarrow H^i(X, \mathcal{F})$ f.d. k-vsp., zero if $i > \dim X$.
(b) $\mathcal{F}, \mathcal{G} \in Coh(X) \Rightarrow Ext^i(\mathcal{F}, \mathcal{G})$ —"—
(c) $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^b(X) \Rightarrow Ext^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ —"—

(One proves (a) $\Rightarrow$ (b) $\Rightarrow$ (c))

if X projective, resolve $\mathcal{F}$ by vector bundles

use truncations and triangles

§6 Pull-back

$$f : X \longrightarrow Y \quad \rightsquigarrow \quad f^* : \mathrm{Mod}_{\mathcal{O}_Y} \xrightarrow[\text{exact}]{f^{-1}} \mathrm{Mod}_{f^{-1}\mathcal{O}_Y} \xrightarrow[\text{left exact}]{- \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X} \mathrm{Mod}_{\mathcal{O}_X}$$

$$Lf^*(-) := f^{-1}(-) \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

__Fact:__ $\exists$ enough flat $f^{-1}\mathcal{O}_Y$-modules on X. $\Rightarrow \exists \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_X}$.

(Partially defined) adjoints:
$$f^* : D^-(\mathrm{QCoh}\, Y) \longrightarrow D^-(\mathrm{QCoh}\, X)$$
$$D^+(\mathrm{QCoh}\, X) \longleftarrow D^+(\mathrm{QCoh}\, X) : f_*$$

Similarly: adjunction
$$\mathcal{F} \overset{\mathbb{L}}{\otimes} - : D^-(\mathrm{QCoh}\, X) \longrightarrow D^-(\mathrm{QCoh}\, X)$$
$$D^+(\mathrm{QCoh}\, X) \longleftarrow D^+(\mathrm{QCoh}\, X) : \mathcal{H}om(\mathcal{F}, -) \quad (\mathcal{F} \text{ coherent})$$

__Fact:__ If X projective scheme (or regular + separated) then there are "enough vector bundles", or X has the "resolution property", that is, given $\mathcal{F} \in \mathrm{Coh}\, X$, there $\exists$ resolution

$$\cdots \longrightarrow \mathcal{E}_2 \to \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F}$$

where the $\mathcal{E}_i$ are vector bundles (i.e. locally free of finite rank)

__Prop 3.26__ If X regular, then every $\mathcal{F}$ has a __finite__ flat resolution (or locally free if X also projective).

__Cor:__ X regular, $\mathcal{F} \in \mathrm{Coh}\, X$. Then $\mathcal{F} \overset{\mathbb{L}}{\otimes} - : D^b(X) \rightleftarrows D^b(X) : R\mathcal{H}om(\mathcal{F}, -)$

__Cor:__ If $f : X \longrightarrow Y$, Y regular, then $Lf^* : D^b(Y) \longrightarrow D^b(X)$.
If also f proper: $Lf_* : D^b(Y) \rightleftarrows D^b(X) : Rf_*$.

§7 Support

closed b/c finite union

Def: $\mathcal{F}^\bullet \in D^b(X)$. $\mathrm{Supp}(\mathcal{F}^\bullet) := \bigcup_i \overbrace{\underbrace{\mathrm{Supp}(\mathcal{H}^i(\mathcal{F}^\bullet))}_{}}^{}$

closed b/c $\mathcal{H}^i(\mathcal{F}^\bullet)$ coherent

Ex: $\mathrm{Supp}(\mathcal{F}[i]) = \mathrm{Supp}(\mathcal{F})\ \forall i$.

Lemma: $f: X \to Y$. $\mathrm{Supp}(Lf^*\mathcal{F}^\bullet) = f^{-1}(\mathrm{Supp}\,\mathcal{F}^\bullet)$

proof: Let $Z = \mathrm{Supp}(\mathcal{F}^\bullet)$, $U = Y \setminus Z$. Then $\mathcal{F}^\bullet|_U$ is acyclic

$\Rightarrow Lf^*\mathcal{F}^\bullet|_{f^{-1}(U)} = Lf|_U^*\mathcal{F}^\bullet|_U$ is also acyclic

$\Rightarrow \mathrm{Supp}(Lf^*\mathcal{F}^\bullet) \subseteq f^{-1}(Z)$.

To show opposite inclusion: Suppose $y \in Z$. Then $\exists m$ maximal s.th. $\mathcal{H}^m(\mathcal{F}^\bullet)_y \neq 0$.

After restricting to $V = Y \setminus \bigcup_{i>m} \mathrm{Supp}\,\mathcal{H}^i$ can assume

$\mathcal{H}^i = 0\ \forall i > m$. Then $\mathcal{H}^m(Lf^*\mathcal{F}^\bullet) = f^*\mathcal{H}^m(\mathcal{F}^\bullet)$ (Exc.)

So enough to prove $\mathrm{Supp}(f^*\mathcal{F}) = f^{-1}(\mathrm{Supp}\,\mathcal{F})$ for $\mathcal{F}$ coherent sheaf. Follows from:

Fact: $\mathcal{F} \in \mathrm{Coh}\,X$ $\quad x \in \mathrm{Supp}(\mathcal{F}) \overset{\text{def}}{\iff} \underbrace{\mathcal{F}_x}_{\text{stalk}} \neq 0 \overset{\text{Nakayama's lemma}}{\iff} \mathcal{F}(x) \neq 0$

fiber $\mathcal{F}(x) := i_x^*\mathcal{F}$

$i_x : \mathrm{Spec}\,k(x) \to X$

Cor: $\mathcal{F}^\bullet \in D^b(X)$. $x \in \mathrm{Supp}(\mathcal{F}^\bullet) \iff \mathcal{F}(x) := Li_x^*\mathcal{F} \neq 0$.

(not acyclic)

__Ex:__ $X = \mathbb{A}^1 = \operatorname{Spec} k[x]$, $\quad \mathcal{F}^\bullet = \Big(\underset{0}{\overset{\widetilde{k[x]}}{\overset{\shortparallel}{\mathcal{O}_X}}} \xrightarrow{\ \cdot x\ } \underset{1}{\overset{\widetilde{k[x]}}{\overset{\shortparallel}{\mathcal{O}_X}}} \Big) \qquad \mathcal{H}^i(\mathcal{F}^\bullet) = \begin{cases} k[x]/(x) & i=1 \\ 0 & \text{o/w} \end{cases}$

$\operatorname{Supp} \mathcal{F}^\bullet = \{0\}$

$$\mathcal{F}^\bullet(0) := i_0^* \mathcal{F}^\bullet = \big(k \xrightarrow{\ 0\ } k \big) \qquad \mathcal{H}^i(\mathcal{F}^\bullet(0)) = \begin{cases} k & i=0,1 \\ 0 & \text{o/w} \end{cases}$$

$$\underset{a \neq 0}{\mathcal{F}^\bullet(a)} = \big(k \xrightarrow{\ \cdot a\ } k \big) \simeq 0$$

__Lemma 3.9:__ $\mathcal{F}^\bullet \in D^b(X)$, $\operatorname{Supp} \mathcal{F}^\bullet = Z_1 \amalg Z_2$ where Z_i closed and disjoint. Then $\mathcal{F}^\bullet = \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet$, $\operatorname{Supp} \mathcal{F}_i^\bullet = Z_i$.

__Proof:__ Induction on length of $\mathcal{F}^\bullet$ $\Big(\underbrace{0 \to \mathcal{F}^m \to \mathcal{F}^{m+1} \to \cdots \to \mathcal{F}^{m+\ell} \to 0}_{\text{has length } \ell} \Big)$

__Exc:__ True when length $=0$, i.e., $\mathcal{F}^\bullet = \mathcal{H}[i]$.

For length >0, let m minimal s.th. $\mathcal{H} := \mathcal{H}^m(\mathcal{F}^\bullet) \neq 0$.
Then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ w/ $\operatorname{Supp}(\mathcal{H}_i) \subseteq Z_i$ (this is length 0 case)

$$\begin{tikzcd} 0 \arrow[r] & \overset{\ker d^m}{\overset{\shortparallel}{\mathcal{H}}} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathcal{F}^m \arrow[r, "d^m"] & \mathcal{F}^{m+1} \arrow[r] & \cdots \end{tikzcd} \quad \rightsquigarrow \quad \text{triangle} \quad \underset{\tau^{\leq m} \mathcal{F}^\bullet}{\underset{\shortparallel}{\mathcal{H}[-m]}} \longrightarrow \mathcal{F}^\bullet \longrightarrow \underset{\tau^{\geq m+1} \mathcal{F}^\bullet}{\underset{\shortparallel}{\mathcal{G}^\bullet}} \longrightarrow \mathcal{H}[-m+1]$$

Since $\mathcal{G}^\bullet$ has shorter length, $\mathcal{G}^\bullet = \mathcal{G}_1^\bullet \oplus \mathcal{G}_2^\bullet$ by induction.

Next we see that induced maps $\mathcal{G}_1^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow \mathcal{H}[-m+1] \longrightarrow \mathcal{H}_2[-m+1]$
$\mathcal{G}_2^\bullet \longrightarrow$ ——"—— $\mathcal{H}_1[-m+1]$

are zero. This follows from:

Lemma: $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in D^b(X)$. If $\mathrm{Supp}(\mathcal{F}_1^\bullet) \cap \mathrm{Supp}(\mathcal{F}_2^\bullet) = 0$ then $R\mathcal{H}om(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) = 0$ and $\mathrm{Hom}(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) = 0$.

proof: Let $Z_i = \mathrm{Supp}(\mathcal{F}_i^\bullet)$ and $U_i = X \setminus Z_i$. Then $X = U_1 \cup U_2$ open covering and

$$R\mathcal{H}om(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)|_{U_i} = R\mathcal{H}om(\mathcal{F}_1^\bullet|_{U_i}, \mathcal{F}_2^\bullet|_{U_i})$$

$$= R\mathcal{H}om(\mathcal{F}_1^\bullet|_{U_1}, 0) = 0 \quad \text{if } i=1$$

$$= R\mathcal{H}om(0, \mathcal{F}_2^\bullet|_{U_2}) = 0 \quad \text{if } i=2$$

Local-to-global spectral sequence:

$$\mathrm{RHom} = R\Gamma \circ R\mathcal{H}om = 0$$

uses $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ flabby if $\mathcal{G}$ flabby. $\square$

So our triangle becomes $\mathcal{H}[-m] \to \mathcal{F}^\bullet \to$

$$\begin{matrix} \mathcal{G}_1^\bullet & \longrightarrow & \mathcal{H}_1[-m+1] \\ \oplus & & \oplus \\ \mathcal{G}_2^\bullet & \longrightarrow & \mathcal{H}_2[-m+1] \end{matrix}$$

$$\Rightarrow \mathcal{F}^\bullet \overset{\text{qis}}{\simeq} \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet \quad \text{where}$$

$$\mathcal{F}_i^\bullet = \mathrm{cone}(\mathcal{G}_i^\bullet[-1] \to \mathcal{H}_i[-m]) \qquad \square$$