Derived Categories March 19th

<u>Def</u> $\mathcal{F}^\bullet \in D^b(X)$

cohomology sheaves

$$\operatorname{supp} \mathcal{F}^\bullet = \bigcup \operatorname{supp} \mathcal{H}^i(\mathcal{F}^\bullet)$$

<u>Lem</u> (H. 3.9)

$\mathcal{F}^\bullet \in D^b(X)$ $\quad$ $\operatorname{Supp} \mathcal{F}^\bullet = Z_1 \sqcup Z_2$

disjoint union of closed subset

Then $\quad \mathcal{F}^\bullet = \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet \quad \operatorname{supp} \mathcal{F}_i = Z_i$

<u>Proof</u> by induction on the length of $\mathcal{F}^\bullet$

$$i_{max}\, \mathcal{F}^\bullet = \max \{ k \mid \mathcal{H}^k(\mathcal{F}^\bullet) \neq 0 \}$$

$$i_{min}\, \mathcal{F}^\bullet = \min \{ \text{———} \}$$

$$\text{length } \mathcal{F}^\bullet = i_{max} - i_{min} + 1$$

length 1 $\mathcal{F}^\bullet \simeq \mathcal{F}[-i]$

$i = i_{max} = i_{min}$

$\mathcal{F} \in \mathrm{Coh}(X)$

→ Trivial from algebraic geometry.

length $\mathcal{F}^\bullet = m \geq 2$

$i := i_{min}(\mathcal{F}^\bullet)$ $\mathcal{H} := \mathcal{H}^i(\mathcal{F}^\bullet)$

$\mathrm{Supp}\,\mathcal{H} \subseteq \mathrm{supp}\,\mathcal{F}^\bullet = Z_1 \cup Z_2$

$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ $\mathrm{supp}\,\mathcal{H}_i \subseteq Z_i$

Now there is a morphism

$$\underset{\mathcal{H}_1 \oplus \mathcal{H}_2[-i]}{\mathcal{H}[-i]} \longrightarrow \mathcal{F}^\bullet \longrightarrow \underset{\mathcal{G}_1^\bullet \oplus \mathcal{G}_2^\bullet}{\mathcal{G}^\bullet} \longrightarrow \underset{(\cdot)\ \mathcal{H}_1 \oplus \mathcal{H}_2[1-i]}{\mathcal{H}[1-i]}$$

induces id on $\mathcal{H}^i(\quad)$ is 0 otherwise

$\mathcal{H}^q(\mathcal{G}^\cdot) = \mathcal{H}^q(\mathcal{F})$ for all $q > i$

$\mathcal{H}^q(\mathcal{G}^\cdot) = 0$ for all $q \leq i$

length $\mathcal{G}^\cdot \leq m-1$

$$\mathcal{G}^\cdot = \mathcal{G}_1^\cdot \cup \mathcal{G}_2^\cdot \qquad \operatorname{supp} \mathcal{G}_i \subseteq Z_i$$

$$E_2^{p,q} := \operatorname{Hom}(\mathcal{H}^{-q}(\mathcal{G}_1^\cdot), \mathcal{H}_2[p]) = 0$$

$$\Downarrow$$

$$\operatorname{Hom}(\mathcal{G}_1^\cdot, \mathcal{H}_2[p+q]) \quad 0$$

the triangle splits and $\mathcal{F}^\cdot \simeq \mathcal{F}_1^\cdot \oplus \mathcal{F}_2^\cdot$

$$\mathcal{H}_j[-i] \longrightarrow \mathcal{F}_j^\cdot \longrightarrow \mathcal{G}_j^\cdot \xrightarrow{+1} \qquad \operatorname{supp} \mathcal{F}_j \subseteq Z_j$$

$\mathcal{T}$ triangulated is decomposable if

$$\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2 \rangle \qquad \mathcal{T}_i \hookrightarrow \mathcal{T} \text{ admissible.}$$

$$\text{No } \mathrm{Hom}(T_i\ T_j) \qquad T_i \in \mathcal{T}_i \qquad i \neq j$$

$$T \in \mathcal{T} \implies T = T_1 \oplus T_2 \qquad T_i \in \mathcal{T}_i$$

<u>Prop</u> $D^b(X)$ is <u>indecomposable</u> $\iff$ X is connected.

<u>Proof</u>

$\Rightarrow$) (The <u>lemma</u> $\Rightarrow$ <u>the</u> contrapositive)

$\Leftarrow$) X is connected by contradiction

$$D^b(X) = \langle D_1 ; D_2 \rangle$$

$\mathcal{O}_X \in D^b(X)$

$\mathcal{O}_X = \mathcal{I}_1^\bullet \oplus \mathcal{I}_2^\bullet \qquad \mathcal{I}_i \in \mathcal{D}_i$

up to quis $\mathcal{I}_i \in \mathrm{Coh}(X) \qquad \mathcal{I}_i \hookrightarrow \mathcal{O}_X$

$\mathcal{I}_i = \mathcal{J}_{X_i}$

$\mathcal{O}_X = \mathcal{J}_{X_1} \oplus \mathcal{J}_{X_2} \subseteq \mathcal{J}_{X_1 \cap X_2} \qquad X_1 \cap X_2 = \emptyset$

$\mathcal{J}_{X_1} \cup \mathcal{J}_{X_2} \subseteq \mathcal{J}_{X_1} \cap \mathcal{J}_{X_2} = 0 \implies X_1 \cup X_2 = X$

contradict X is connected

$\mathcal{I}_1 = \mathcal{O}_X \qquad \mathcal{O}_X \in \mathcal{D}_1$

$a \in X$

$k(x) = K_1 \oplus K_2$ either K_1 or $K_2 = 0$

$\mathcal{O}_X \twoheadrightarrow k(x)$ in $\mathrm{Coh}(X)$

$$\mathrm{Hom}_{D^b(X)}(\mathcal{O}_x, k(x)) \neq 0$$

$$k(x) \in D_1$$

We show $D_2 = 0$

$\mathcal{F}^\bullet \in D_2$ $\quad \mathcal{F}^\bullet \neq 0$

$i := i_{max}(\mathcal{F}^\bullet)$ $\quad \mathcal{H}^i = \mathcal{H}^i(\mathcal{F}^\bullet)$

$x \in \mathrm{Supp}(\mathcal{H}^i)$

$$\mathcal{H} \twoheadrightarrow k(x)$$

$$\mathcal{H}^i[-i] \to k(x)[-i]$$

$\mathcal{F} \underset{q}{\simeq} \tau_{\leq i}(\mathcal{F}^\bullet) \to \mathcal{H}^i[-i]$ $\quad \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, k(x)) \neq 0$

$$\mathcal{F}^{i-1} \to \ker d^i \to 0$$

ξ.

Serre Functors

$\mathcal{C}$ $\quad S: \mathcal{C} \longrightarrow \mathcal{C}$ is called (a)
Serre functor if there are functorial isos

$$\eta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, S(X))^{\vee}$$

For every $X \in \mathcal{C}$ $\quad S(X)$ represents

$$Y \longmapsto \mathrm{Hom}(X,Y)^{\vee}$$

$X \rightsquigarrow S(X)$ unique up to unique isom
(Yoneda)

$\Rightarrow$ A Serre functor is unique up to unique functorial isomorphism.

Thm

1) Serre functors are exact (triangulated)

2) They commute with exact (Δ) equivalence.

$$\Phi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$$

$$\begin{array}{ccc} S_1 & & S_2 \end{array}$$

$$\Phi \circ S_1 \simeq S_2 \circ \Phi$$

Exactness:

$$\mathrm{Hom}(Y, \underline{S(X)[1]})^\vee \cong \mathrm{Hom}(Y[-1], S(X))^\vee$$

$$\cong \mathrm{Hom}(X, Y[-1])$$

$$\cong \mathrm{Hom}(X[1], Y)$$

$$\cong \mathrm{Hom}(Y, \underline{S(X[1])})^\vee$$

$$S(X)[1] \cong S(X[1])$$

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

$$\mathrm{Hom}(X, S(A)) \to \mathrm{Hom}(X, S(B)) \to \mathrm{Hom}(X, S(C))$$

$$\to \mathrm{Hom}(X, S(A)[1]) \to \ldots$$

exact

$$\begin{tikzcd} S(C) \arrow[r, "S(h)"] & S(A)[1] \arrow[r] & E \arrow[r] \arrow[d, mapsto] & S(C)[1] \\ & & c(S(h)) & \end{tikzcd}$$

$$\begin{tikzcd} S(C)[-1] \arrow[r] \arrow[d, equal] & S(A) \arrow[r] \arrow[d, equal] & E[-1] \arrow[r] \arrow[d, dashed, "\varphi"] & S(C) \arrow[d, equal] \\ S(C)[-1] \arrow[r] & S(A) \arrow[r] & S(B) \arrow[r] & S(C) \end{tikzcd}$$

Construct φ using the exact sequence

5 lemma $\longrightarrow$ iso

(usual

E rep $\mathrm{Hom}(B, Y)^\vee \Longrightarrow E - S(B)$

Commutativity. $A, B \in \mathcal{C}_1 \quad \Phi : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$

$$\mathrm{Hom}(A, S(B))^{\vee} \simeq \mathrm{Hom}(\Phi(A), \Phi S_1(B))^{\vee}$$

$$\begin{array}{c} \mathrm{Hom}(A, S(B))^{\vee} \\ \wr\wr \\ \mathrm{Hom}(B, A) = \mathrm{Hom}(\Phi(B), \Phi(A)) \end{array}$$

$X \in \mathcal{C}_2, \quad Y \in \mathcal{C}_2$

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(\Phi(B), \Phi(A))$$

for some A, B

$$\simeq \mathrm{Hom}(\Phi(A), \Phi(S_1(B)))^{\vee}$$

$$\simeq \mathrm{Hom}(Y, \underline{\Phi S_1(B)})^{\vee}$$

Represent the same thing rep by

$S_2(X) = S_2 \Phi(B)$

Back to Geometry

X smooth projective

$$S_X(?) := (?) \otimes \omega_X [\dim X]$$

is a Serre functor.

$$\omega_X = \Lambda^n \Omega_X$$

$$\eta_{\mathcal{E}^\bullet \mathcal{F}^\bullet} : \mathrm{Hom}(\mathcal{E}^\bullet \mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X [\dim X])^\vee$$

$f : X \longrightarrow Y$ morphism of smooth scheme

$$\dim f = \dim X - \dim Y$$

$$\omega_f = \omega_X \otimes \omega_Y^* \qquad \omega_{X/Y}$$

Prop $\mathcal{F},\mathcal{G} \in \mathrm{Coh}(X)$ X smooth proj.

$$\mathrm{Ext}^i(\mathcal{F},\mathcal{G}) = 0 \quad \forall\, i > \dim X$$

(homological dimension of $\underline{\mathrm{Coh}(X) = n}$)

Proof $n = \dim X$

$$\mathrm{Ext}^n(\mathcal{O}_X, \omega_X) \cong \mathrm{Ext}^n(\mathcal{O}_X, \mathcal{O}_X)^\vee = k^\vee \neq 0$$

Main idea is to use Serre duality to describe Ext^n as "negative exts" which are 0.

$$Rf_* R\underline{\mathcal{H}om}(\mathcal{F}^\bullet, Lf^* \mathcal{E}^\bullet \underline{\otimes} \omega_f [\dim f])$$

$$\simeq R\underline{\mathcal{H}om}(Rf_* \mathcal{F}^\bullet, \mathcal{E}^\bullet)$$

Recovers Serre when $f: X \longrightarrow \operatorname{Spec} k$.

$$\mathbb{D}_X(?) = R\mathcal{H}om(?, \omega_X[\dim X])$$

$$Rf_* \circ \mathbb{D}_X \simeq \mathbb{D}_Y \circ Rf_*$$

$$f^{!\bullet} := \mathbb{D}_X Lf^* \mathbb{D}_Y^{-1}$$

<u>COR</u> X smooth projective $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(X)$

$$R\mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \in D^b(\mathrm{Ab})$$

$$\mathrm{Ext}^p(\mathcal{E}^\bullet, \mathcal{H}^q(\mathcal{F}^\bullet)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$$

Need

$A \in \mathcal{A}$ $\quad RF(A) \in D^b(\mathcal{B})$ $\quad F: \mathcal{A} \longrightarrow \mathcal{B}$ left exact

C smooth curve ($\mathcal{A}$ abelian hom dim $\mathcal{A} = 1$)

$D^b(\mathcal{A}) \ni \mathcal{F}^\bullet \cong \bigoplus \mathcal{E}_i[-i]$

$\Phi : D^b(X) \longrightarrow D^b(Y)$ equivalence
(X, Y smooth)

then $\dim X = \dim Y$

then ω_X and ω_Y have same order
in $\mathrm{Pic}(X)$ / $\mathrm{Pic}(Y)$

$$S_X = (\ ) \otimes \omega_X [\dim X]$$
$$S_Y = (\ ) \otimes \omega_Y [\dim Y]$$

$x \in X$ $\quad k(x)$ is "special"

$$k(x) \otimes \omega_X \simeq k(x)$$

$$\phi(k(x)) = \phi(k(x) \otimes \omega_X) = \phi(S_X(k(x))[-\dim X])$$
$$S_Y(\phi(k(x)))[\dim X] = \Phi(k(x)) \otimes \omega_Y [\dim Y - \dim X]$$

$$i = i_{max}(\phi(k(x))) \;/\; i_{min}\,\phi(k(x))$$

$$0 \neq \mathcal{H}^i(\phi(k(x))) = \mathcal{H}^i(\phi(k(x)) \otimes \omega_Y[\dim Y - \dim X])$$

$$= \mathcal{H}^{\underbrace{i + \dim Y - \dim X}}(\phi(k(x)) \otimes \omega_Y)$$

if $\omega_X^k \simeq \mathcal{O}_X$ $\qquad S_X^k(?) \simeq (?)[k \dim X]$

$$\phi^{-1} S_Y^k[k \dim Y] \circ \phi = id_{D^b(X)}$$

$$S_Y^k[k \dim Y] = \phi\, id_{D^b(X)}\, \phi^{-1} = id_{D^b(Y)}$$

$$\Longleftarrow \quad \omega_Y^k = \mathcal{O}_X$$