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Nonlinear Algebra

Goals: Don't be afraid of algebra!

Identifying algebraic structures is helpful.

Today: Projective space is your friend

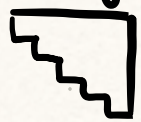
Example 1: LU factorization

$$A = L\,U$$

square matrix

lower triangular

upper triangular



Not every A has such a factorization!

Can you find a 2×2 example?

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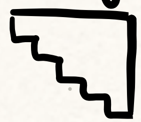

Not every A has such a factorization!

Exercise: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an LU decomposition iff $a \neq 0$ or $b = 0$ or $c = 0$.

Can you find a 2×2 example?

Example 1: LU factorization

In other words, the image of

$$\varphi_{LU} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^{2\times 2}$$

$$\left(\begin{bmatrix} u & \\ v & w \end{bmatrix}, \begin{bmatrix} x & y \\ & z \end{bmatrix} \right) \longmapsto \begin{bmatrix} ux & uy \\ vx & vy+wz \end{bmatrix}$$

is

$$\left\{ \begin{bmatrix} * & * \\ * & * \end{bmatrix} \right\} \setminus \left(\left\{ \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} \right\} \setminus \left(\left\{ \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \right\} \right) \right).$$

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This is <u>not</u> closed (in Euclidean topology)!

Its closure is $\mathbb{R}^{2\times 2}$, i.e.,

every A can be approximated by an LU decomposition.

Example 2: Linear Convolutional Networks (LCNs)

A convolution on 1-dimensional signals is a linear map given by a Toeplitz matrix

$$
\left[\begin{array}{ccc|cccccccc}
w_0 & w_1 & \cdots & w_s & \cdots & w_{k-1} & & & & & \\
 & & & w_0 & w_1 & \cdots & \cdots & w_{k-1} & & & \\
 & & & & \ddots & \ddots & & & \ddots & & \\
 & & & & & w_0 & w_1 & \cdots & \cdots & w_{k-1} &
\end{array}\right]
$$

$$\underbrace{\qquad\qquad}_{s}$$

with filter $w \in \mathbb{R}^k$ and stride s.

An LCN (single channel, 1-dim. signals) composes several such linear maps, one for each layer.

Example 2: Linear Convolutional Networks (LCNs)

$$\begin{bmatrix} w_0 & w_1 \end{bmatrix} \cdot \begin{bmatrix} v_0 & v_1 & v_2 & & \\ & & v_0 & v_1 & v_2 \end{bmatrix} = \begin{bmatrix} w_0 v_0 & w_0 v_1 & w_0 v_2 + w_1 v_0 & w_1 v_1 & w_1 v_2 \end{bmatrix}$$

layer 2:
filter $w \in \mathbb{R}^2$
stride 2.

layer 1:
filter $v \in \mathbb{R}^3$
stride 2

Does every $A \in \mathbb{R}^5$ have such a factorization?

Can it be approximated by one?

Example 2: Linear Convolutional Networks (LCNs)

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layer 2: filter $w \in \mathbb{R}^2$ stride 2.

layer 1: filter $v \in \mathbb{R}^3$ stride 2

Does every $A \in \mathbb{R}^5$ have such a factorization?

No!

Can it be approximated by one?

No!

Exercise: $A = \begin{bmatrix} a & b & c & d & e \end{bmatrix}$ has such a factorization iff $ad^2 + b^2e = bcd$ & $c^2 \geq 4ae$.

The set of such A is closed (in Euclidean topology)!

$$\varphi_{LU} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^{2\times 2}$$

image is <u>not</u> closed

$$\varphi_{LCN} : \mathbb{R}^2 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^5$$

$$(w, v) \longmapsto \begin{bmatrix} w_0 & w_1 \end{bmatrix} \cdot \begin{bmatrix} v_0 & v_1 & v_2 & & \\ & & v_0 & v_1 & v_2 \end{bmatrix}$$

image is closed

$$\boxed{\text{What is the difference ?}}$$

$\varphi_{LU}: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^{2\times 2}$

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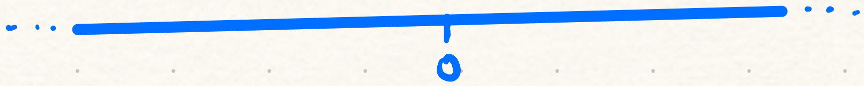
image is closed

$\boxed{\text{What is the difference?}}$

φ_{LU} is <u>not</u> a projective morphism

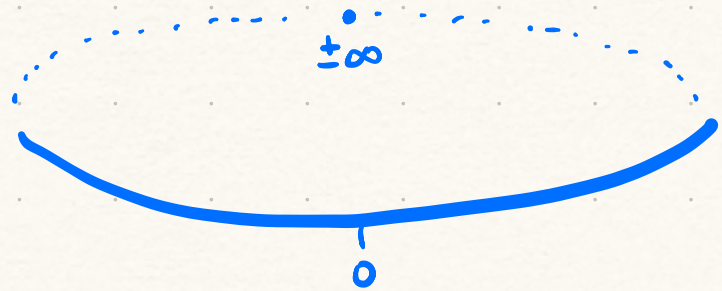
φ_{LCN} is a projective morphism

Issue: Affine space is not compact



affine line $\mathbb{R}^1$

Projective space is compact



projective line $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R}^1 \cup \{\pm\infty\}$

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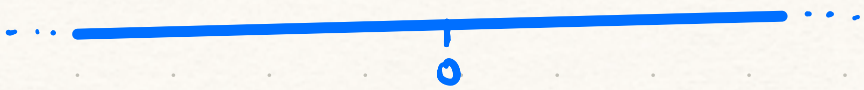

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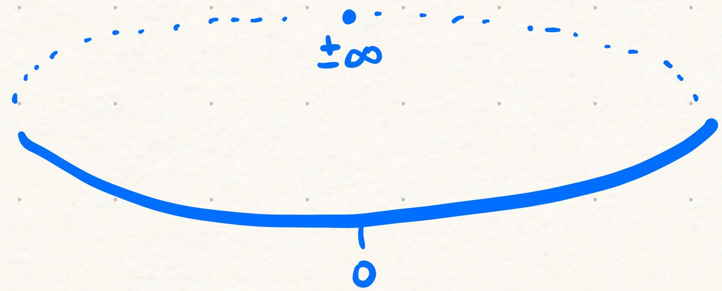
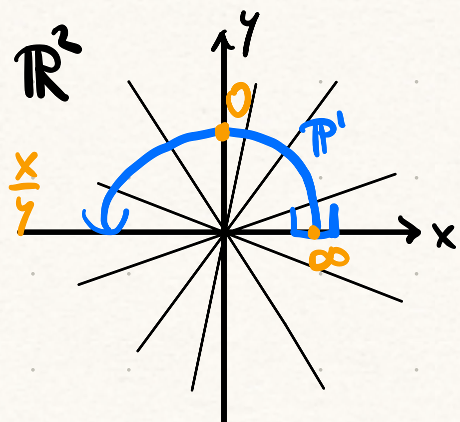


projective line $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R}^1 \cup \{\pm\infty\}$

standard construction:

$\mathbb{P}^n_{\mathbb{R}} = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$, where

$u \sim v \;:\Leftrightarrow\; \exists \lambda \in \mathbb{R}\setminus\{0\} : u = \lambda v$



$\mathbb{P}^n_{\mathbb{R}}$ = set of lines in $\mathbb{R}^{n+1}$ through origin

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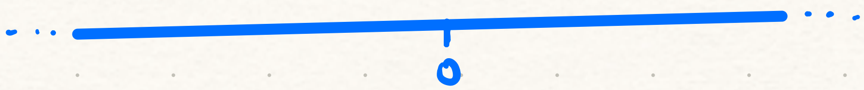
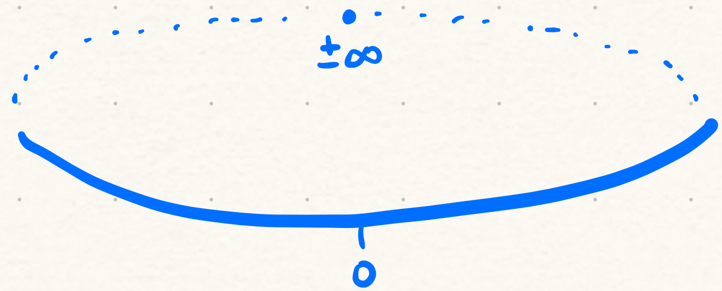


affine line $\mathbb{R}^1$

Projective space is compact ☺



projective line $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R}^1 \cup \{\pm\infty\}$

$\mathbb{P}^n_{\mathbb{R}}$ is the sphere in $\mathbb{R}^{n+1}$ after identifying antipodal points

compact

$\Rightarrow$ $\mathbb{P}^n_{\mathbb{R}}$ is compact in the quotient topology of the Euclidean topology on $\mathbb{R}^{n+1}$

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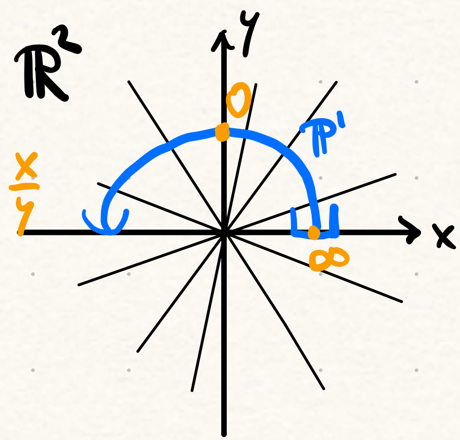

$\mathbb{P}^n_{\mathbb{R}}$ = set of lines in $\mathbb{R}^{n+1}$ through origin

Standard fact from topology:

$\varphi : T_1 \to T_2$ continuous map between topological spaces,
T_1 compact
$\Longrightarrow$ $\mathrm{im}(\varphi)$ is compact

Standard fact from topology:

$$\boxed{\begin{array}{l} \varphi : T_1 \to T_2 \text{ continuous map between topological spaces,} \\ \quad T_1 \text{ compact} \\ \Longrightarrow \ \mathrm{im}(\varphi) \text{ is compact} \end{array}}$$

not compact ☹

$$\varphi_{LU} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^{2\times 2}$$

$$\left(\begin{bmatrix} u & \\ v & w \end{bmatrix}, \begin{bmatrix} x & y \\ & z \end{bmatrix} \right) \longmapsto \begin{bmatrix} u & \\ v & w \end{bmatrix} \cdot \begin{bmatrix} x & y \\ & z \end{bmatrix} = \begin{bmatrix} ux & uy \\ vx & vy+wz \end{bmatrix}$$

continuous ☺

Can we projectivize φ_{LU}?

$$\varphi_{LU}(\alpha L, \beta U) = \alpha L \cdot \beta U = \alpha\beta\, LU = \alpha\beta\, \varphi_{LU}(L, U),$$

so φ_{LU} is well-behaved under scaling the factors and it makes sense to consider

$$\tilde{\varphi}_{LU} : \mathbb{P}^2_{\mathbb{R}} \times \mathbb{P}^2_{\mathbb{R}} \longrightarrow \mathbb{P}^3_{\mathbb{R}}$$

$$\left(\left[\begin{matrix} u & \\ v & w \end{matrix}\right], \left[\begin{matrix} x & y \\ & z \end{matrix}\right] \right) \longmapsto \left[\begin{matrix} u & \\ v & w \end{matrix} \cdot \begin{matrix} x & y \\ & z \end{matrix} \right] = \left[\begin{matrix} ux & uy \\ vx & vy + wz \end{matrix} \right]$$

defined up to scaling!

BUT ...

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defined up to scaling!

BUT $\tilde{\varphi}_{LU}$ is not defined everywhere!

E.g. $\begin{matrix} 0 & \\ 0 & 1 \end{matrix} \cdot \begin{matrix} 2 & 3 \\ & 0 \end{matrix} = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \notin \mathbb{P}^3_{\mathbb{R}}$

cannot apply topology fact!

Exercise: Compute all points in $\mathbb{P}^2_{\mathbb{R}} \times \mathbb{P}^2_{\mathbb{R}}$, where $\tilde{\varphi}_{Ln}$ is not defined.

This is called the base locus of $\tilde{\varphi}_{Ln}$.

Standard fact from topology:

$\varphi : T_1 \to T_2$ continuous map between topological spaces,
T_1 compact
$\Longrightarrow$ $\mathrm{im}(\varphi)$ is compact

$$\varphi_{LCN} : \mathbb{R}^2 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^5$$
$$(w, v) \longmapsto \begin{bmatrix} w_0 & w_1 \end{bmatrix} \cdot \begin{bmatrix} v_0 & v_1 & v_2 & & \\ & & v_0 & v_1 & v_2 \end{bmatrix}$$

can be projectivized: $\tilde{\varphi}_{LCN} : \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^2_{\mathbb{R}} \longrightarrow \mathbb{P}^4_{\mathbb{R}}$
$$[w], [v] \longmapsto \left[\begin{bmatrix} w_0 & w_1 \end{bmatrix} \cdot \begin{bmatrix} v_0 & v_1 & v_2 & & \\ & & v_0 & v_1 & v_2 \end{bmatrix} \right]$$

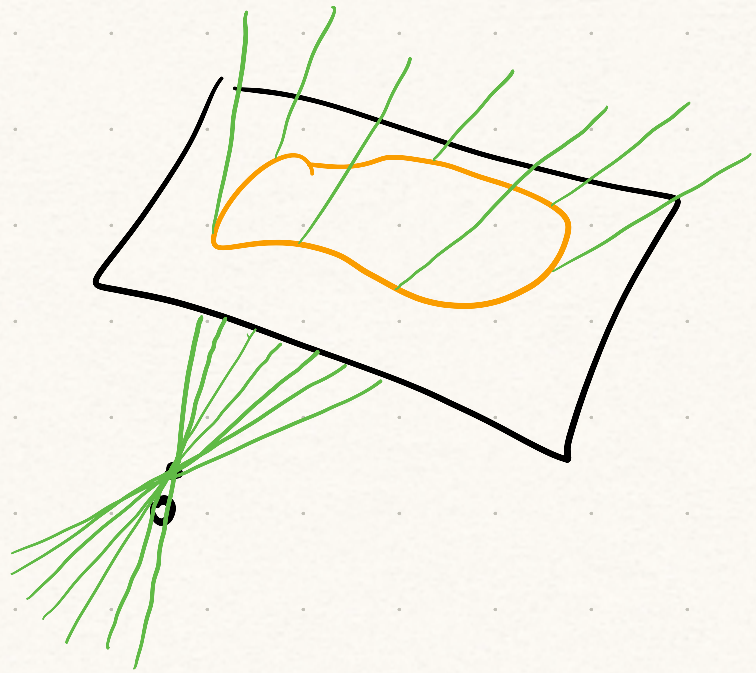
Exercise: $\tilde{\varphi}_{LCN}$ is defined everywhere, i.e., its base locus is empty.

can apply topology fact

Hence: $\mathrm{im}(\tilde{\varphi}_{LCN}) \subseteq \mathbb{P}^4_{\mathbb{R}}$ is compact (in Euclidean quotient top.).

Since $\mathrm{im}(\varphi_{LCN}) \subseteq \mathbb{R}^5$ is the affine cone over $\mathrm{im}(\tilde{\varphi}_{LCN})$, it is closed (in Euclidean topology).

replace each projective point by the affine line it represents



Another example: Low-rank matrix completion

Which $\begin{bmatrix} & b \\ c & d \end{bmatrix}$ can be completed to a rank-1 matrix?

Equivalently, what is the image of

$$\varphi_1 : \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, ad = bc \right\} \longrightarrow \mathbb{R}^3,$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (b, c, d)$$

?

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Exercise: $\operatorname{im}(\varphi_1) = \{(b,c,d) \mid d \neq 0 \text{ or } b = 0 \text{ or } c = 0\}$

<u>not</u> closed in Euclidean topology

What is the image of

$$\varphi_2 : \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, ad = bc \right\} \longrightarrow \mathbb{R}^3,$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (b, c, a-d)$$

?

What is the image of

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?

Exercise: $\operatorname{im}(\varphi_2) = \left\{ (b, c, \Delta) \;\middle|\; \frac{\Delta^2}{4} + bc \geq 0 \right\}$

is closed in Euclidean topology

Topology reason:

φ_2 is well-behaved under scaling (i.e., $\varphi_2(\lambda A) = \lambda \varphi_2(A)$)

and can be projectivized:

$$\tilde{\varphi}_2 : \overbrace{\left\{ \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \in \mathbb{P}^3_{\mathbb{R}} \;\middle|\; ad = bc \right\}}^{\text{projective variety} \;\Rightarrow\; \text{compact} \;☺} \longrightarrow \mathbb{P}^2_{\mathbb{R}}$$

$$\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \longmapsto [b, c, a-d]$$

$\tilde{\varphi}_2$ is defined everywhere ☺

(since only $\lambda \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right]}_{\text{not of rank } 1}$ maps to $(0,0,0)$)

$$\tilde{\varphi}_2 : \underbrace{\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}^3_{\mathbb{R}} \;\middle|\; ad = bc \right\}}_{\text{projective variety} \Rightarrow \text{compact} \ \smile} \longrightarrow \mathbb{P}^2_{\mathbb{R}}$$

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$\tilde{\varphi}_2$ is defined everywhere $\smile$

(since only $\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ maps to $(0,0,0)$)
not of rank 1

$\xRightarrow[\text{fact}]{\text{topology}}$ $\operatorname{im}(\tilde{\varphi}_2)$ is compact

$\Rightarrow \operatorname{im}(\varphi_2)$ is closed in Euclidean topology

We cannot apply the topology fact to

$$\tilde{\varphi}_1 : \overbrace{\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}^3_{\mathbb{R}} \;\middle|\; ad = bc \right\}}^{\text{projective variety} \;\Rightarrow\; \text{compact}} \longrightarrow \mathbb{P}^2_{\mathbb{R}}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto [b, c, d]$$

since it is not defined at $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

rank 1

Definition: Algebraic Varieties

- $\mathbb{K}$ field, often $\mathbb{R}$ or $\mathbb{C}$ (or $\mathbb{Q}$ or a finite field for computations)

- An affine variety is a subset of $\mathbb{K}^n$ that is the solution set of a system of polynomial equations.

- $\mathbb{P}^n_{\mathbb{K}} := (\mathbb{K}^{n+1} \setminus \{0\})/\sim$ projective space $[u \sim v :\Leftrightarrow \exists \lambda \in \mathbb{K} \setminus \{0\} : u = \lambda v]$

- A projective variety is a subset of $\mathbb{P}^n_{\mathbb{K}}$ that is the solution set of a system of homogeneous polynomial equations.
  all terms have the same degree

Example: $f := xy - z$ is inhomogeneous

$f(1,1,1) = 0 \neq f(2,2,2)$,

although $[1,1,1] = [2,2,2] \in \mathbb{P}^2$

$g := xy - z^2$ is homogeneous

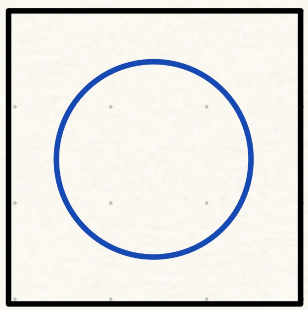
$g(\lambda x, \lambda y, \lambda z) = \lambda^2 g(x,y,z)$

☺

Important

Projective varieties are compact in the Euclidean topology, affine varieties are generally not (only closed).

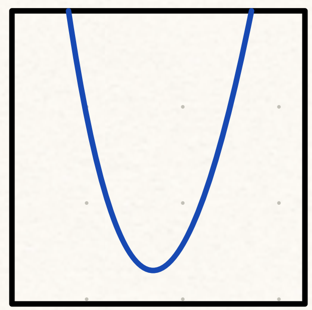
Example: $\mathbb{R}^2$



$x^2+y^2-1=0$

compact in $\mathbb{R}^2$

$\mathbb{R}^2$



$y-x^2=0$

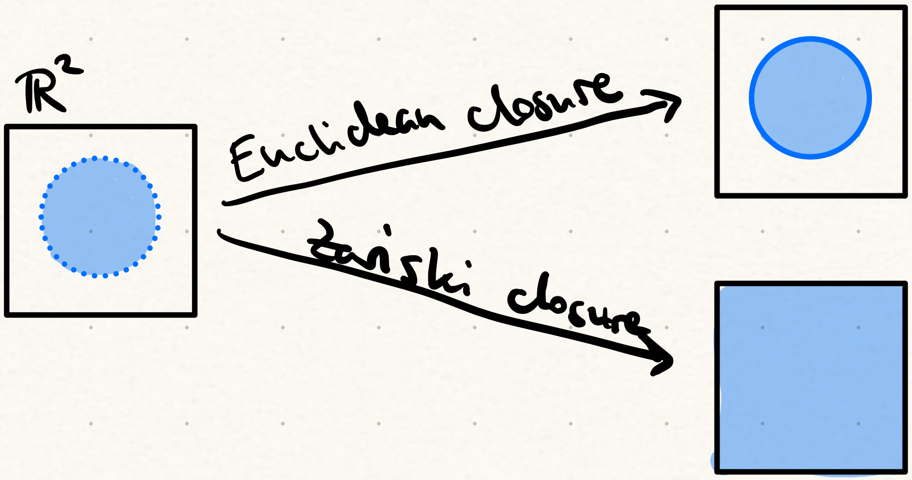
not compact in $\mathbb{R}^2$

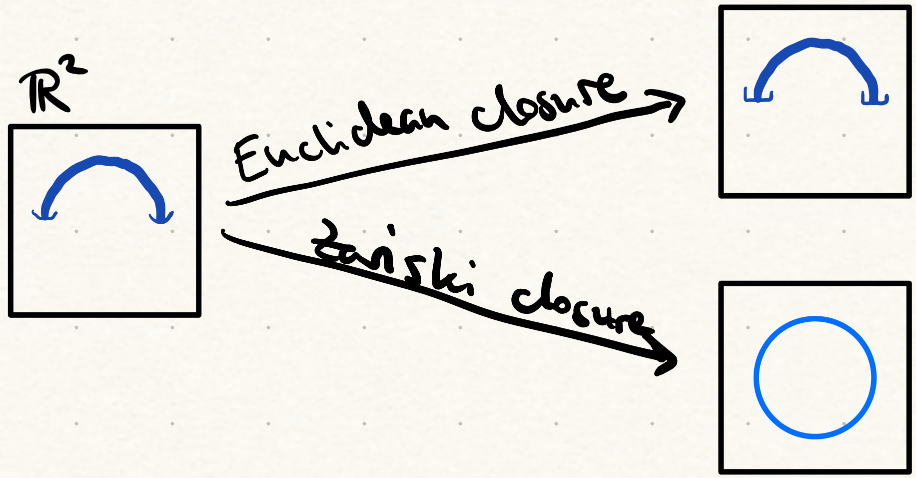
Definition: Zariski topology

Zariski topology on $\mathbb{K}^n$ resp. $\mathbb{P}^n_{\mathbb{K}}$: closed sets are the algebraic varieties

The Zariski topology is coarser than the Euclidean topology:

- Zariski closed implies Euclidean closed
- Zariski open implies Euclidean open





What are the Zariski closed sets in $\mathbb{R}^1$?

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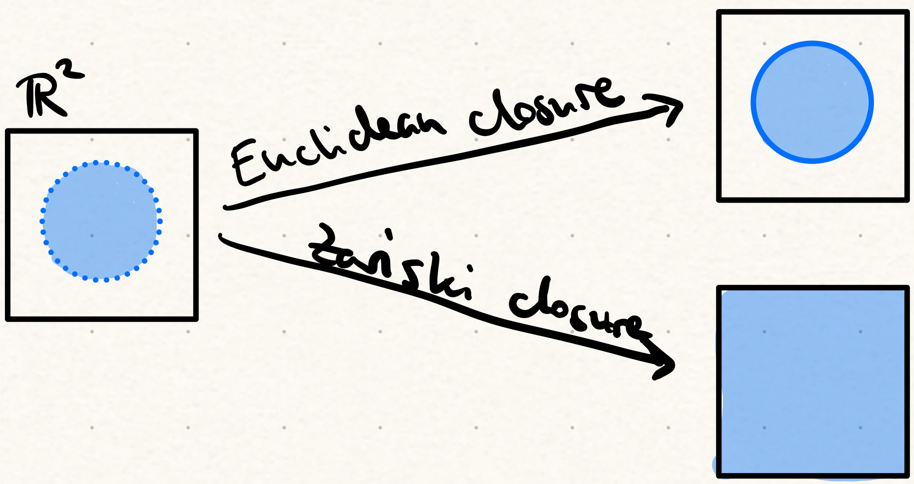

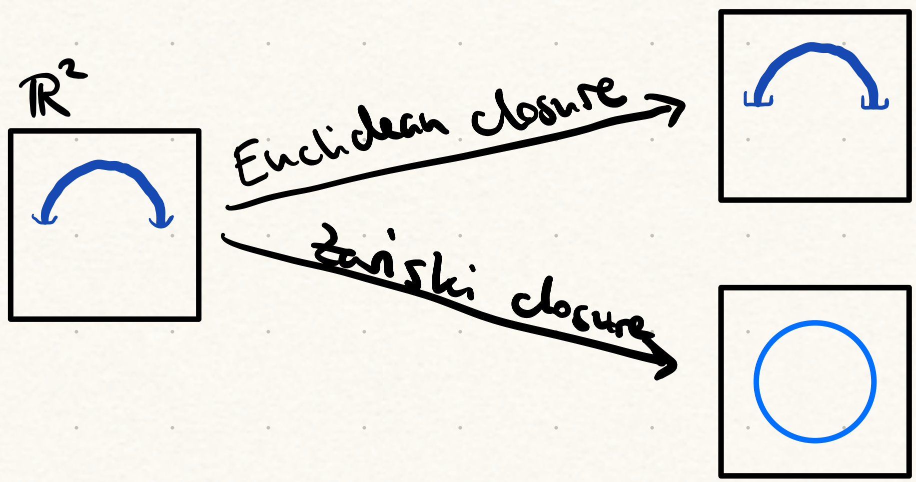

What are the Zariski closed sets in $\mathbb{R}^1$?

What are the Zariski closed sets in $\mathbb{R}^2$?

$\mathbb{R}^1$, finitely many points, $\emptyset$

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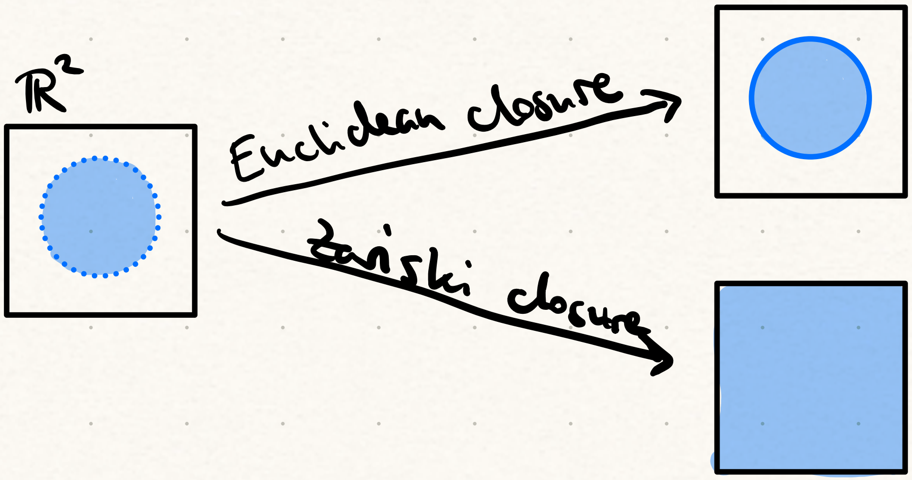
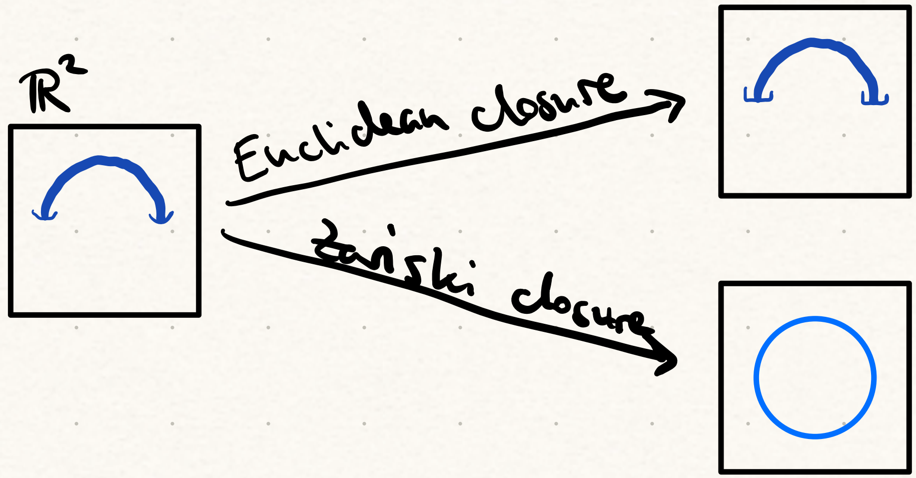


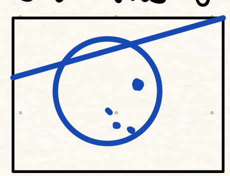


What are the Zariski closed sets in $\mathbb{R}^1$?

What are the Zariski closed sets in $\mathbb{R}^2$?

$\mathbb{R}^1$, finitely many points, $\emptyset$

$\mathbb{R}^2$, finite unions of algebraic curves & points, $\emptyset$



Definition: Maps

- A morphism between affine varieties $\underset{\mathbb{K}^m}{\overset{}{X}} \xrightarrow{\varphi} Y$ with $X \subseteq \mathbb{K}^m$, $Y \subseteq \mathbb{K}^n$, is a polynomial map, i.e., of the form $\varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))$ with $\varphi_i \in \mathbb{K}[x_1, \dots, x_m]$.

- A polynomial map between projective varieties $X \xrightarrow{\varphi} Y$ needs all coordinate functions φ_i to be homogeneous of the same degree.

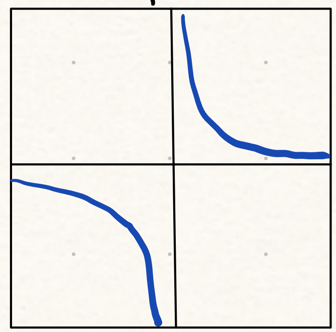
If φ is defined on all of X, it is a projective morphism. topology fact applies ☺

Otherwise, it is called a rational map and we write $X \dashrightarrow Y$.

The base locus of φ is the set of $x \in X$ where φ is not defined.

Definition: Semi-Algebraic Sets

$\mathbb{R}^2$



$xy = 1$

An algebraic set / algebraic variety in $\mathbb{R}^n$ is a solution set of a system of polynomial equations in $\mathbb{R}[x_1, \dots, x_n]$.

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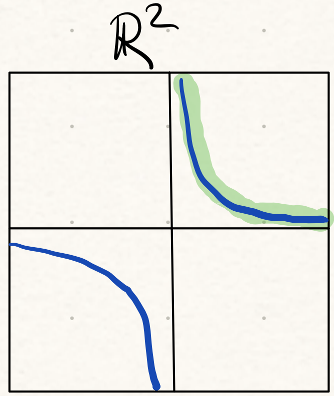


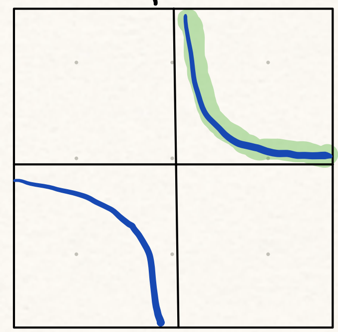
$xy = 1$

$x > 0$

A **basic semialgebraic set** in $\mathbb{R}^n$ is a solution set of a system of polynomial equations and polynomial inequalities in $\mathbb{R}[x_1,\dots,x_n]$.

Definition: Semi-Algebraic Sets

$\mathbb{R}^2$



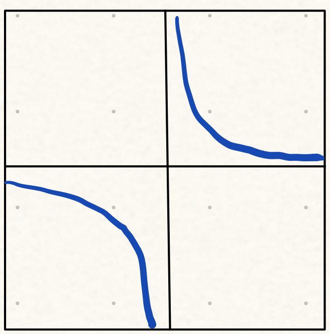
$xy = 1$

$x > 0$

An **algebraic set / algebraic variety** in $\mathbb{R}^n$ is a solution set of a system of polynomial equations in $\mathbb{R}[x_1, \ldots, x_n]$.

A **basic semialgebraic set** in $\mathbb{R}^n$ is a solution set of a system of polynomial equations and polynomial inequalities in $\mathbb{R}[x_1, \ldots, x_n]$.

A **semialgebraic set** is a finite union of basic semialgebraic sets.



$xy = 1$

$$\longrightarrow$$

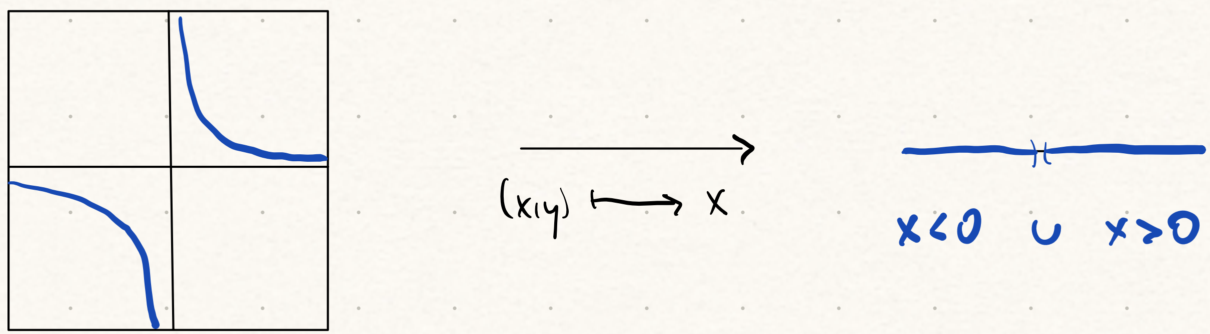
$(x,y) \longmapsto x$

$x < 0 \cup x > 0$

Tarski-Seidenberg Theorem

A **morphism between algebraic varieties** $\varphi : \underset{\mathbb{R}^n}{\overset{\cup\!\shortmid}{X}} \longrightarrow \underset{\mathbb{R}^m}{\overset{\cup\!\shortmid}{Y}}$ is a polynomial map, i.e., $\varphi = (\varphi_1, \dots, \varphi_m)$ & $\varphi_i \in \mathbb{R}[x_1, \dots, x_n]$.

Thm: $\varphi(X)$ is a semialgebraic set.



$xy=1$

$\varphi: X \to Y$ morphism between varieties

$\mathrm{im}(\varphi)$	Ex.: φ_{LU} affine varieties	Ex.: φ_{LCN} projective varieties
$\mathbb{K} = \mathbb{R}$		Euclidean closed
$\mathbb{K} = \mathbb{C}$		Euclidean closed

$\varphi: X \to Y$ morphism between varieties

$\mathrm{im}(\varphi)$	Ex.: φ_{LLL} affine varieties	Ex.: φ_{LCN} projective varieties
$\mathbb{K} = \mathbb{R}$	semi-algebraic = finite union of solution sets of polynomial equations & polynomial inequalities Tarski-Seidenberg theorem	semi-algebraic Euclidean closed
$\mathbb{K} = \mathbb{C}$		Euclidean closed

$\varphi: X \to Y$ morphism between varieties

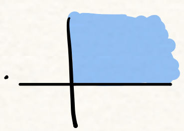
$\operatorname{im}(\varphi)$	affine varieties (Ex.: φ_{LU})	projective varieties (Ex.: φ_{LCN})
$\mathbb{K} = \mathbb{R}$	semi-algebraic = finite union of solution sets of polynomial equations & polynomial inequalities Tarski-Seidenberg theorem	semi-algebraic Euclidean closed
$\mathbb{K} = \mathbb{C}$	constructible = finite union of differences of varieties Chevalley's theorem	Euclidean closed

Example: $\left\{\boxed{\begin{matrix} * & * \\ * & * \end{matrix}}\right\} \setminus \left(\left\{\begin{bmatrix} 0 & * \\ * & * \end{bmatrix}\right\} \setminus \left(\left\{\begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}\right\} \cup \left\{\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}\right\}\right)\right)$

$= \left(\left\{\boxed{\begin{matrix} * & * \\ * & * \end{matrix}}\right\} \setminus \left\{\begin{bmatrix} 0 & * \\ * & * \end{bmatrix}\right\}\right) \cup \left\{\begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}\right\} \cup \left\{\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}\right\}$

$\varphi: X \to Y$ morphism between varieties

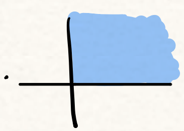
$\mathrm{im}(\varphi)$	Ex.: φ_{LU} affine varieties	Ex.: φ_{LCN} projective varieties
$\mathbb{K} = \mathbb{R}$	semi-algebraic = finite union of solution sets of polynomial equations & polynomial inequalities Tarski-Seidenberg theorem	semi-algebraic Euclidean closed
$\mathbb{K} = \mathbb{C}$	constructible = finite union of differences of varieties Chevalley's theorem	Euclidean closed

Over $\mathbb{C}$: The Euclidean closure of a constructible set is a variety
(i.e., Euclidean closure = Zariski closure for constructible sets).
This is in general false for semialgebraic sets, e.g.  $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$.

$\varphi: X \to Y$ morphism between varieties

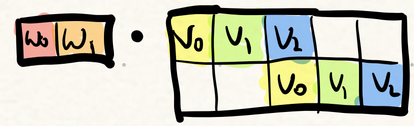
$\mathrm{im}(\varphi)$	affine varieties (Ex.: φ_{LLU})	projective varieties (Ex.: φ_{LCN})
$\mathbb{K} = \mathbb{R}$	semi-algebraic = finite union of solution sets of polynomial equations & polynomial inequalities Tarski-Seidenberg theorem	semi-algebraic Euclidean closed
$\mathbb{K} = \mathbb{C}$	constructible = finite union of differences of varieties Chevalley's theorem	variety (Euclidean closed) main theorem of elimination theory

Over $\mathbb{C}$: The Euclidean closure of a constructible set is a variety (i.e., Euclidean closure = Zariski closure for constructible sets).

This is in general false for semialgebraic sets, e.g.  $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$.

$\varphi_{LU}: \mathbb{K}^3 \times \mathbb{K}^3 \longrightarrow \mathbb{K}^{2\times 2}$

$\varphi_{LCN}: \mathbb{K}^2 \times \mathbb{K}^3 \longrightarrow \mathbb{K}^5$

$(w, v) \longmapsto$ 

<u>$\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$:</u>

$\mathrm{im}(\varphi_{LU}) =$

$$\left\{\begin{bmatrix} * & * \\ * & * \end{bmatrix}\right\} \setminus \left\{\begin{bmatrix} 0 & * \\ * & * \end{bmatrix}\right\} \cup \left\{\begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}\right\} \cup \left\{\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}\right\}$$

constructible

semi-algebraic over $\mathbb{R}$

not Euclidean closed

<u>$\mathbb{K} = \mathbb{R}$:</u>

$$\mathrm{im}(\varphi_{LCN}) = \left\{ \boxed{a|b|c|d|e} \;\middle|\; ad^2 + b^2e = bcd \;\&\; c^2 \geq 4ae \right\}$$

semi-algebraic Euclidean closed

<u>$\mathbb{K} = \mathbb{C}$:</u>

$$\mathrm{im}(\varphi_{LCN}) = \left\{ \boxed{a|b|c|d|e} \;\middle|\; ad^2 + b^2e = bcd \right\}$$

variety