Week 1 – Warm Up

September 2, 2026

Instructions: Do Exercise 1, Exercise 2, ... today. The exercises marked with a *, you can do today if you have time and find them interesting; if the * exercises sound too technical / boring, just try to understand the statements and do them at any later point in this class, once we have seen more fun things.

1 Varieties & Ideals

This class develops tools to investigate and solve systems of multivariate polynomial equations (and inequalities), with a particular focus on systems arising in engineering applications. Key players in this class are **algebraic varieties**.

Definition 1. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ denote the field of real or complex numbers. An **algebraic variety** in $\mathbb{K}^n$ is the solution set of polynomial equations in n variables.

We denote by $\mathbb{K}[x_1, \ldots, x_n]$ the set of all polynomials in the variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$. (Note: This set is a *ring*.)

Exercise 1. *What is a (commutative) ring? Why is $\mathbb{K}[x_1, \ldots, x_n]$ a commutative ring?*

Exercise 2. *Let $\mathcal{F} \subseteq \mathbb{K}[x_1, \ldots, x_n]$ and let*

$$Z(\mathcal{F}) := \{p \in \mathbb{K}^n \mid \forall f \in \mathcal{F} : f(p) = 0\}$$

be the solution set of $\mathcal{F}$. Show that the points in $Z(\mathcal{F})$ are also solutions of:

a) $f_1 + f_2$ for all $f_1, f_2 \in \mathcal{F}$, and

b) gf for all $g \in \mathbb{K}[x_1, \ldots, x_n]$, $f \in \mathcal{F}$.

Hence, if we are interested in solving a given set of polynomials, we might as well extend that set by the operations in Exercise 2a) and 2b), until we obtain a so-called **ideal**.

Definition 2. A nonempty subset $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is called an **ideal** if the following two conditions hold:

a) If $f_1, f_2 \in I$, then $f_1 + f_2 \in I$;

b) If $g \in \mathbb{K}[x_1, \dots, x_n]$ and $f \in I$, then $gf \in I$.

Exercise 3. *Let $S \subseteq \mathbb{K}^n$ be an arbitrary subset and let*

$$I(S) := \{f \in \mathbb{K}[x_1, \dots, x_n] \mid \forall p \in S : f(p) = 0\}$$

denote the set of all polynomials that vanish on S. Show that $I(S)$ is an ideal.

Definition 3. Given a subset $S \subseteq \mathbb{K}^n$, $I(S)$ is the **vanishing ideal** of S.

The following exercise shows that all varieties arise as solution sets of ideals (i.e., we do not need to consider systems of polynomial equations that are not ideals).

Exercise 4. *Let $\mathcal{F} \subseteq \mathbb{K}[x_1, \dots, x_n]$ and let $Z := Z(\mathcal{F})$ be its solution set. Show that:*

a) $\mathcal{F} \subseteq I(Z)$;

b) $Z(I(Z)) = Z$.

In fact, for any set $\mathcal{F} \subseteq \mathbb{K}[x_1, \dots, x_n]$ of polynomials, we can consider the smallest ideal in $\mathbb{K}[x_1, \dots, x_n]$ containing $\mathcal{F}$. We denote this ideal by $\langle \mathcal{F} \rangle$ and call it the **ideal generated by** $\mathcal{F}$. We then have that $\mathcal{F} \subseteq \langle \mathcal{F} \rangle \subseteq I(Z(\mathcal{F}))$ and will see in the following that $Z(\langle \mathcal{F} \rangle) = Z(\mathcal{F})$. To formally construct the ideal $\langle \mathcal{F} \rangle$, we proceed as follows:

Exercise* 5. *Given two ideals $I, J \subseteq \mathbb{K}[x_1, \dots, x_n]$, show that their intersection $I \cap J$ is also an ideal.*

Exercise* 6. *Given $\mathcal{F} \subseteq \mathbb{K}[x_1, \dots, x_n]$, consider the set*

$$\mathcal{I}_{\mathcal{F}} := \{I \subseteq \mathbb{K}[x_1, \dots, x_n] \text{ ideal } \mid \mathcal{F} \subseteq I\}$$

of ideals containing $\mathcal{F}$. Show that $\langle \mathcal{F} \rangle := \bigcap_{I \in \mathcal{I}_{\mathcal{F}}} I$ is an ideal.

Exercise* 7. *Let $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathbb{K}[x_1, \dots, x_n]$. Show that $Z(\mathcal{G}) \subseteq Z(\mathcal{F})$.*

Exercise* 8. *Let $\mathcal{F} \subseteq \mathbb{K}[x_1, \dots, x_n]$. Conclude from Exercises 4b) and 7 that $Z(\langle \mathcal{F} \rangle) = Z(\mathcal{F})$.*

The remarkable **Hilbert's Basis Theorem** states that every ideal in $\mathbb{K}[x_1, \dots, x_n]$ is generated by a finite set $\mathcal{F}$ of polynomials.

Take-away: *It is always sufficient to consider* ***finite*** *systems of polynomial equations. Moreover, it is totally equivalent to study solution sets of finitely many polynomial equations or of ideals in the ring of polynomials (which perspective you prefer, is just a matter of taste).*

2 Projective Varieties

Next week, we will see that our favorite varieties are projective ones. The **n-dimensional projective space** is constructed as follows: We call two nonzero vectors u and v in $\mathbb{K}^{n+1}$ *equivalent*, denoted by $u \sim v$, if one is a scalar multiple of the other, i.e., there is some nonzero $\lambda \in \mathbb{K}$ such that $u = \lambda v$.

Definition 4. The **n-dimensional projective space** is the set of equivalence classes of the equivalence relation $\sim$:

$$\mathbb{P}^n_{\mathbb{K}} := (\mathbb{K}^{n+1} \setminus \{0\}) / \sim .$$

In simpler terms, $\mathbb{P}^n_{\mathbb{K}}$ is the set of nonzero vectors up to scaling (by a multiplicative constant).

Also in projective spaces we can consider solutions sets of polynomials, as long as the polynomials are **homogeneous**.

Definition 5. Let $d \in \mathbb{Z}_{\geq 0}$. A *monomial of degree* d in the variables $x_0, \ldots, x_n$ is of the form $x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n}$, where $i_0, i_1, \ldots, i_n \in \mathbb{Z}_{\geq 0}$ and $i_0 + i_1 + \ldots + i_n = d$. A polynomial in $\mathbb{K}[x_0, \ldots, x_n]$ is **homogeneous of degree** d if it is a $\mathbb{K}$-linear combination of monomials of degree d.

Note that the zero polynomial is considered homogeneous of any degree.

Exercise 9. *Let $f \in \mathbb{K}[x_0, \ldots, x_n]$ be homogeneous of some degree, and let $p \in \mathbb{K}^{n+1}$, $\lambda \in \mathbb{K} \setminus \{0\}$. Show that: $f(p) = 0 \iff f(\lambda p) = 0$.*

The previous exercise shows that, given a point $\mathbf{p} \in \mathbb{P}^n_{\mathbb{K}}$ in projective space and a homogeneous polynomial f, the condition $f(\mathbf{p}) = 0$ is well-defined (i.e., it holds either for all or for none of the points on the line in $\mathbb{K}^{n+1}$ that $\mathbf{p}$ represents). Therefore, given homogeneous polynomials, we can study their solution sets in projective space:

Definition 6. Let $\mathcal{F} \subseteq \mathbb{K}[x_0, \ldots, x_n]$ be a set of homogeneous polynomials (potentially of different degrees). Its solution set in projective space

$$Z_{\mathbb{P}}(\mathcal{F}) := \{\mathbf{p} \in \mathbb{P}^n_{\mathbb{K}} \mid \forall f \in \mathcal{F} : f(\mathbf{p}) = 0\}$$

is called **projective variety**.

In contrast, the varieties in Definition 1 are often called **affine varieties**. Moreover, given a set $\mathcal{F} \subseteq \mathbb{K}[x_0, \ldots, x_n]$ of homogeneous polynomials, the affine variety $Z(\mathcal{F}) \subseteq \mathbb{K}^{n+1}$ is called the **affine cone** over the projective variety $Z_{\mathbb{P}}(\mathcal{F}) \subseteq \mathbb{P}^n_{\mathbb{K}}$; it is obtained by replacing each projective point by the affine line it represents.

Finally, we note that homogeneity is necessary for the construction of projective varieties:

Exercise 10. *If $f \in \mathbb{K}[x_0, \ldots, x_n]$ is not homogeneous, why is then the condition $f(\mathbf{p}) = 0$ not well-defined?*

3 Images of Polynomial Maps

Let us consider a very simple low-rank matrix completion problem: Which partial matrices $\begin{bmatrix} & b \\ c & d \end{bmatrix}$ can be completed to a rank-one matrix?

Note that the set of 2×2 matrices of rank at most one is an algebraic variety, as it is the solution set of the determinant:

$$\mathcal{M}_1^{2\times 2} := \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | ad - bc = 0 \}.$$

We can either view it as an affine variety in $\mathbb{K}^{2\times 2} \cong \mathbb{K}^4$ or as a projective variety in $\mathbb{P}^3_{\mathbb{K}}$, as the determinant is homogeneous. As before, $\mathbb{K}$ is either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$.

Now, our low-rank matrix completion problem can be reformulated as follows: Determine the image of the map

$$\varphi_1 : \mathcal{M}_1^{2\times 2} \longrightarrow \mathbb{K}^3, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (b, c, d).$$

Exercise 11. *Show that* $\operatorname{im}(\varphi_1) = \{(b, c, d) \mid d \neq 0 \text{ or } b = 0 \text{ or } c = 0\}$.

Note that this image is not closed (in the Euclidean topology) and that its closure is all of $\mathbb{K}^3$. This means that every partial matrix can be approximated by rank-one completable matrices, but not every partial matrix is exactly completable.

The behavior changes drastically if we take different linear measurements of a given rank-one matrix. Instead of recording the entries b, c, d, we now measure $b, c, a - d$.

Exercise 12. *Show that the image of*

$$\varphi_2 : \mathcal{M}_1^{2\times 2} \longrightarrow \mathbb{K}^3, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (b, c, a - d)$$

is

a) all of $\mathbb{C}^3$ *if* $\mathbb{K} = \mathbb{C}$, *and*

b) $\{(b, c, \Delta) \in \mathbb{R}^3 \mid \Delta^2 + 4bc \geq 0\}$ *if* $\mathbb{K} = \mathbb{R}$.

The image of φ_2 is closed (in the Euclidean topology). Over the complex numbers, every linear measurement can be completed to a matrix of rank at most one. Over the real numbers, this is not true, and measurements that cannot be completed can also not be approximated by completable measurements.

Exercise 13. *Discuss: What has changed? Do you have explanations for this change in behavior?*

Here is another pair of examples that undergoes the same change in behaviour. The following map encodes the LU factorization of a 2×2 matrix:

$$\begin{aligned} \varphi_{\mathrm{LU}} : \mathbb{K}^3 \times \mathbb{K}^3 &\longrightarrow \mathbb{K}^{2\times 2}, \\ ((u,v,w),(x,y,z)) &\longmapsto \begin{bmatrix} u & 0 \\ v & w \end{bmatrix} \cdot \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} ux & uy \\ vx & vy+wz \end{bmatrix}. \end{aligned}$$

Exercise* 14. *Show:* $\operatorname{im}(\varphi_{\mathrm{LU}}) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a \neq 0 \textit{ or } a = b = 0 \textit{ or } a = c = 0 \}$.

Moreover, the next map encodes a very small instance of a *linear convolutional neural network*:

$$\begin{aligned} &\varphi_{\mathrm{LCN}} : \mathbb{K}^2 \times \mathbb{K}^3 \longrightarrow \mathbb{K}^5, \\ &((u,v),(x,y,z)) \longmapsto \begin{bmatrix} u & v \end{bmatrix} \cdot \begin{bmatrix} x & y & z & 0 & 0 \\ 0 & 0 & x & y & z \end{bmatrix} = \begin{bmatrix} ux & uy & uz+vx & vy & vz \end{bmatrix}. \end{aligned}$$

Exercise* 15. *Show that* $\operatorname{im}(\varphi_{\mathrm{LCN}})$ *equals*

a) $\{ \begin{bmatrix} a & b & c & d & e \end{bmatrix} \mid ad^2 + b^2 e = bcd \}$ *if* $\mathbb{K} = \mathbb{C}$, *and*

b) $\{ \begin{bmatrix} a & b & c & d & e \end{bmatrix} \mid ad^2 + b^2 e = bcd,\ c^2 \geq 4ae \}$ *if* $\mathbb{K} = \mathbb{R}$.

Preview: *Next week, we will learn some criteria to distinguish between these different behaviors. Throughout the class, we will hands-on investigate and solve algebraic optimization problems and algebraic inverse problems, coming from various fields of engineering (robotics, computer vision, deep learning, etc.). Examples of algebraic optimization problems include finding a closest low-rank tensor and training algebraic neural networks. Instances of algebraic inverse problems are computing exact tensor decompositions and 3D-scene reconstruction from 2D-images. We will make use of both theoretical insights from basic algebraic geometry and software computations.*