Week 2 – Exercises

September 9, 2026

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Given a nonzero vector $(p_0, p_1, \ldots, p_n) \in \mathbb{K}^{n+1}$, we write $[p_0 : p_1 : \ldots : p_n] \in \mathbb{P}^n_{\mathbb{K}}$ for the corresponding point in projective space.

1 Base Locus

Exercise 1. *Let* $\mathcal{M}_1^{2\times 2} = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}^3_{\mathbb{K}} \mid ad - bc = 0\}$ *be the set of rank-one* 2×2 *matrices up to scaling, and consider the following maps* $\tilde{\varphi}_i : \mathcal{M}_1^{2\times 2} \to \mathbb{P}^2_{\mathbb{K}}$*:*

$$\tilde{\varphi}_1 : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto [b : c : d],$$
$$\tilde{\varphi}_2 : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto [b : c : a - d].$$

Show that the base locus of $\tilde{\varphi}_1$ *is a single point, while the base locus of* $\tilde{\varphi}_2$ *is empty.*

Exercise 2. *Show that the base locus of the following map is empty:*

$$\begin{aligned} &\tilde{\varphi}_{\mathrm{LCN}} : \mathbb{P}^1_{\mathbb{K}} \times \mathbb{P}^2_{\mathbb{K}} \longrightarrow \mathbb{P}^4_{\mathbb{K}}, \\ &([u : v], [x : y : z]) \longmapsto \begin{bmatrix} u & v \end{bmatrix} \cdot \begin{bmatrix} x & y & z & 0 & 0 \\ 0 & 0 & x & y & z \end{bmatrix} = [ux : uy : uz + vx : vy : vz]. \end{aligned}$$

Exercise 3. *Compute the base locus of the following map:*

$$\begin{aligned} &\tilde{\varphi}_{\mathrm{LU}} : \mathbb{P}^2_{\mathbb{K}} \times \mathbb{P}^2_{\mathbb{K}} \longrightarrow \mathbb{P}^3_{\mathbb{K}}, \\ &([u : v : w], [x : y : z]) \longmapsto \begin{bmatrix} u & 0 \\ v & w \end{bmatrix} \cdot \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} ux & uy \\ vx & vy + wz \end{bmatrix}. \end{aligned}$$

2 Constructible & Semialgebraic Sets

Recall that a constructible set is a finite union of differences of varieties, i.e., it is of the form $\bigcup_{i=1}^k (X_i \setminus Y_i)$, where X_i, X_i are algebraic varieties.

If $\mathbb{K} = \mathbb{R}$, we define a semialgebraic set to be of the form $\bigcup_{i=1}^m A_i$, where each A_i is a solution set of finitely many polynomial equations and inequalities.

Exercise 4. *Let* $\mathbb{K} = \mathbb{R}$.

a) Show that every constructible set is semialgebraic.

b) Give an example of a semialgebraic set that is not constructible.

Exercise 5. *a) Let* $X, Y \subseteq \mathbb{K}^n$ *be algebraic varieties. Prove that* $X \cup Y$ *is an algebraic variety.*

b) Find a polynomial equation system whose solution set is the union of the point $\{(0,0)\}$ *and the line* $\{(x,y) \in \mathbb{K}^2 \mid x + y - 1 = 0\}$.

Exercise 6. *Let* $\mathbb{K} = \mathbb{R}$. *Plot the plane curve* $C \subseteq \mathbb{R}^2$ *defined by* $y^2 = x^2(x-1)$. *Observe that* $(0,0)$ *is a so-called 'isolated singularity' (i.e., in an open Euclidean ball around* $(0,0)$*, the point* $(0,0)$ *is the only point on* C*).*

a) Conclude that the constructible set $C \setminus \{(0,0)\}$ *is closed in the Euclidean topology.*

b) Show that the constructible set $C \setminus \{(0,0)$ *is not closed in Zariski topology. Why?*
(hint: this is a little more difficult to argue formally with the current tools we have; but please discuss this question with your peers ***and*** *ask an LLM)*

The following two problems show that the behavior observed in Exercise 6 is very different over the complex numbers:

Exercise 7. *Let* $\mathbb{K} = \mathbb{C}$. *Consider the plane curve* $C \subseteq \mathbb{C}^2$ *defined by* $y^2 = x^2(x-1)$ *(same as in Exercise 6). Show that the Euclidean closure of the constructible set* $C \setminus \{(0,0)\}$ *equals* C.

Exercise 8 (difficult)**.** *Let* $\mathbb{K} = \mathbb{C}$. *Show that the Euclidean closure of a constructible set in* $\mathbb{C}^n$ *equals its Zariski closure.*

3 Topology Fact

Exercise 9. *Let* $\varphi : T_1 \to T_2$ *be a continuous map between topological spaces and let* T_1 *be compact. Prove that the image of* φ *is compact.*
(hint: you might have to look up some definitions from basic topology)