Solution to exercises on polynomial approximations

1) Quadratic approximation of $\sqrt{4.1}$:

$f(x) = \sqrt{x}$ $\qquad f(4) = 2$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\qquad f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4} x^{-3/2}$ $\qquad f''(4) = -\frac{1}{4}(4)^{-3/2} = -\frac{1}{4}\left((4)^{\frac{1}{2}}\right)^{-3} = -\frac{1}{4}\frac{1}{8} = -\frac{1}{32}$

$$p(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2 =$$
$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$$p(4.1) = 2 + \frac{1}{4}\frac{1}{10} - \frac{1}{64}\frac{1}{100} = \frac{12800+160-1}{6400} = \frac{129590}{6400}$$

2) $f(x) = \ln(1+x)$ $\qquad$ $p/100$ small so we check near 0

$f(0) = \ln(1) = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(x) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\qquad f''(x) = -1$

$$p(x) = (x-0) - \frac{1}{2}(x-0)^2$$
$$p(p/100) = \frac{p}{100} - \frac{1}{2} p/1000 = \frac{20-1}{2000} p = \frac{19}{2000} p$$

3) $f(x) = e^x$ $\qquad$ we have that $f^{(n)}(x) = e^x$ for every n.

$$p_d(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{4!}(x-1)^4$$
$$+ + \frac{e}{d!}(x-1)^d$$

$$= \sum_{i=0}^{d} \frac{e}{i!} (x-1)^i$$

for degree 50 just set $d=50$

- $f(x) = x^2 e^x \qquad x_0 = 1 \qquad$ degree 3

$f(1) = e$

$f'(x) = 2xe^x + x^2e^x = (2x+x^2)e^x \qquad f'(1) = 3e$

$f''(x) = (2x+2)e^x + (2x+x^2)e^x = (x^2+4x+2)e^x \qquad f''(1) = 7e$

$f'''(x) = (2x+4)e^x + (x^2+4x+2)e^x = (x^2+6x+6)e^x \qquad f'''(1) = 13e$

$p(x) = e + 3e(x-1) + \frac{7e}{2}(x-1)^2 + \frac{13e}{6}(x-1)^3$

$p(1.1) = e + \frac{3e}{10} + \frac{7e}{200} + \frac{13e}{6000}$

- $f(x) = \ln(1+e^x) \qquad x=0$

$f(0) = \ln(2)$

$f'(x) = e^x \frac{1}{1+e^x} \qquad f'(0) = \frac{1}{2}$

$f''(x) = e^x \frac{1}{1+e^x} - e^x \cdot e^x \frac{1}{(1+e^x)^2} = e^x \frac{1}{1+e^x} - e^{2x} \frac{1}{(1+e^x)^2}$

$f''(0) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$f'''(x) = e^x \frac{1}{1+e^x} - e^{2x} \frac{1}{(1+e^x)^2} - 2e^{2x} \frac{1}{(1+e^x)^2} + e^{3x} \frac{1}{(1+e^x)^3}$

$= e^x \frac{1}{1+e^x} - 3e^{2x} \frac{1}{(1+e^x)^2} + e^{3x} \frac{\cdot 1}{(1+e^x)^3}$

$f'''(0) = \frac{1}{2} - \frac{3}{4} + \frac{1}{8} = -\frac{1}{8}$

$p(x) = \ln(2) + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3$

$p(0.1) = \ln(2) + \frac{1}{20} + \frac{1}{800} - \frac{1}{4800}$

- $f(x) = \ln(x^2+2x+1) \qquad x_0 = 1 \qquad \deg 3$

$f(1) = \ln(4)$

$f'(x) = (2x+2)\dfrac{1}{x^2+2x+1} \qquad f'(1) = 4\,\dfrac{1}{4} = 1$

$f''(x) = 2\,\dfrac{1}{x^2+2x+1} - (2x+2)\,\dfrac{1}{(x^2+2x+1)^2} \qquad f''(1) = \dfrac{1}{2} - \dfrac{4}{16} = \dfrac{1}{4}$

$f'''(x) = -2(2x+2)\dfrac{1}{(x^2+2x+1)^2} - 2\,\dfrac{1}{(x^2+2x+1)^2} + (2x+2)^2\dfrac{1}{(x^2+2x+1)^3}$

$= -2(2x+1)\dfrac{1}{(x^2+2x+1)^2} + (2x+2)^2\dfrac{1}{(x^2+2x+1)^3}$

$f'''(1) = -6\,\dfrac{1}{16} + 16\,\dfrac{1}{64} = \dfrac{-24+16}{64} = -\dfrac{8}{64} = -\dfrac{1}{8}$

$p(x) = \ln(4) + 1(x-1) + \dfrac{1}{8}(x-1)^2 - \dfrac{1}{48}(x-1)^3$

- $e^{1/10}$ with two correct decimal digit

the error has to be smaller than 0.005

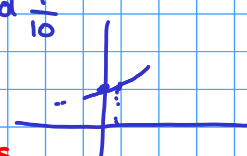
$f(x) = e^x \qquad x_0 = 0$

The error in the Taylor formula is

$$\left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-x_0)^{n+1} \right| = \left| \frac{e^z}{(n+1)!}\left(\frac{1}{10}\right)^{n+1} \right| \quad \text{for some } z$$

in a small interval containing 0 and $\frac{1}{10}$

We can assume $\boxed{e^z \le \frac{1}{2}}$



here one has to be carefull $e^z < 2$ would have been a safer choice

$$\text{error} \le \frac{1}{2(n+1)!}\left(\frac{1}{10}\right)^{n+1} < 0.0005$$

$$\frac{1}{(n+1)!}\left(\frac{1}{10}\right)^{n+1} < 0.001 \qquad (< 0.00025 \text{ with } 2)$$

$n=1 \qquad \dfrac{1}{4}\,\dfrac{1}{100} = \dfrac{1}{400} = 0.0025$ does not work

$n = 2$ $\quad \frac{1}{6} \frac{1}{1000} = 0.0001 < 0.001$

this works for $c^3 < 2$

Thus we can stop at degree 2

$p(x) = 1 + x + \frac{1}{2}x^2$ $\quad p\left(\frac{1}{10}\right) = 1 + \frac{1}{10} + \frac{1}{200} = \frac{221}{200}$

$= 1.105$ $\quad$ check the value of $e^{1/10}$

Recap

① Estimate $f^{(n+1)}(z)$ in the interval containing x_0 and the point you want to evaluate

② Find for which n $\quad \left| \frac{f^{(n+1)}(z)}{(n+1)!} (x_0 - p)^{n+1} \right| <$ what you want

③ compute the taylor polynomial of degree n

④ evaluate in p