Tutorial

Taylor Series – An Overview

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January 16, 2009

Abstract

This document is intended to give a helpful introduction to the idea of Taylor series. First the general specification for one- and two-dimensional real functions are presented. This is followed by some mathematical examples to provide insight into the real applicability of the theorem.

1 General Definition

In mathematics, the *Taylor series* is a representation of a function as an infinite sum of terms calculated from the values of its derivatives at a single point. It may be regarded as the limit of the Taylor polynomials. Taylor series are named in honour of English mathematician Brook Taylor. If the series uses the derivatives at point zero, the series is also called a Maclaurin series, named after Scottish mathematician Colin Maclaurin.

(From Wikipedia: http://en.wikipedia.org/wiki/Taylor_series)

A one-dimensional Taylor series is an expansion of a real function $f(x)$ around a point $x = x_0$ and given by one-dimensional

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{(2!)} f''(x_0) + \dots \\ &\quad + \frac{(x-x_0)^n}{(n!)} f^n(x_0) + R_n \\ &= \sum_{k=0}^{n} \frac{(x-x_0)^k f^{(k)}(x_0)}{(k!)} + R_n. \end{aligned} \tag{1}$$

Here, R_n is a remainder term known as the *Lagrange remainder*. Using the mean value theorem, for some $x \in [x_0, x]$, therefore, integrating $n+1$ times gives the result

$$R_n = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x^*). \tag{2}$$

So the maximum error after n terms of the Taylor series is the maximum value of (2) running through all $x^* \in [x_0, x]$. An alternative form of the one-dimensional Taylor series may be obtained by letting

$$x - x_0 = \Delta x$$

so that

$$x = x_0 + \Delta x.$$

Substitute this result into (1) to give

$$f(x_0 + \Delta x) = f(x_0) + \Delta x f'(x_0) + \frac{1}{2!}(\Delta x)^2 f''(x_0) + \frac{1}{3!}(\Delta x)^3 f'''(x_0) + \ldots \tag{3}$$

A Taylor series of a real function in two variables $f(x, y)$ is given by

two-dimensional

$$\begin{aligned} f(x + \Delta x, y + \Delta y) = \\ & f(x, y) + [f_x(x, y)\Delta x + f_y(x, y)\Delta y] + \\ & \frac{1}{2!}\left[(\Delta x)^2 f_{xx}(x, y) + 2\Delta x\Delta y f_{xy}(x, y) + (\Delta y)^2 f_{yy}(x, y)\right] + \\ & \frac{1}{3!}\left[(\Delta x)^3 f_{xxx}(x, y) + 3(\Delta x)^2\Delta y f_{xxy}(x, y) + 3\Delta x(\Delta y)^2 f_{xyy}(x, y)\right. \\ & \qquad \left. + (\Delta y)^3 f_{yyy}(x, y)\right] + \ldots \end{aligned} \tag{4}$$

This can be further generalized for a real function in n variables,

n-dimensional

$$f(x_1, \ldots, x_n) = \sum_{j=0}^{\infty} \left\{ \frac{1}{j!} \left[\sum_{k=1}^{n} (x_k - a_k) \frac{\partial}{\partial x_k} \right]^j f(x_1, \ldots, x_n) \right\}. \tag{5}$$

Rewriting,

$$f(x_1 + a_1, \ldots, x_n + a_n) = \sum_{j=0}^{\infty} \left\{ \frac{1}{j!} \left[\sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k} \right]^j f(x_1, \ldots, x_n) \right\}. \tag{6}$$

Note that you may need to use the Binomial theorem to solve for higher order terms,

$$\begin{aligned} (x + y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\ &= x^n + \left[\sum_{k=1}^{n-1} \frac{n!}{(n-k)! \cdot k!} x^{n-k} y^k \right] + y^n \end{aligned}$$

For example, taking $n = 2$ in (5) gives

Example $n = 2$

$$\begin{aligned} f(x_1, x_2) &= \sum_{j=0}^{\infty} \left\{ \frac{1}{j!} \left[(x_1 - a_1)\frac{\partial}{\partial x_1}(x_2 - a_2)\frac{\partial}{\partial x_2} \right]^j f(x_1, x_2) \right\} \\ &= f(a_1, a_2) + \left[(x_1 - a_1)\frac{\partial f}{\partial x_1} + (x_2 - a_2)\frac{\partial f}{\partial x_2} \right] + \\ &\quad \frac{1}{2!}\left[(x_1 - a_1)^2 \frac{\partial^2 f}{\partial x_1^2} + 2(x_1 - a_1)(x_2 - a_2)\frac{\partial^2 f}{\partial x_1 \partial x_2} + (x_2 - a_2)^2 \frac{\partial^2 f}{\partial x_2^2} \right] + \ldots. \end{aligned}$$

For a real function with three variables, taking $n = 3$ in (6) gives

Example $n = 3$

$$f(x_1 + a_1, x_2 + a_2, x_3 + a_3) = \sum_{j=0}^{\infty} \left\{ \frac{1}{j!} \left(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3} \right)^j f(x_1, x_2, x_3) \right\}.$$

2 Exemplifying Applications

Example 1

To differentiate or integrate functions of Ito processes, we need to make use of *Ito's Lemma*. Ito's Lemma is easiest understood as a Taylor series expansion. Suppose that $x(t)$ follows the process

$$dx = a(x,t)dt + b(x,t)dz, \tag{7}$$

and consider a function $F(x,t)$ that is at least twice differentiable in x and once in t. We would like to find the total differential of this function, dF. The usual rules of calculus define this differential in terms of first-order changes in x and t:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt. \tag{8}$$

But suppose that we also include higher-order terms for changes in x:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt + \tfrac{1}{2} \frac{\partial^2 F}{\partial x^2} (dx)^2 + \tfrac{1}{6} \frac{\partial^3 F}{\partial x^3} (dx)^3 + \dots. \tag{9}$$

Substituting equation (5) for dx, we see that some terms in dx include dt raised to a power higher than 1, and so will go to zero faster than dt in the limit. Hence Ito's Lemma gives the differential dF as

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt + \tfrac{1}{2} \frac{\partial^2 F}{\partial x^2} (dx)^2. \tag{10}$$

Example 2

Consider a continuous-time stochastic process that yields a simple form for the Bellman equation

$$\rho F(x,t) = \max_u \left\{ \pi(x,u,t) + \frac{1}{dt} E\left[dF\right] \right\}.$$

Allowing the drift and the diffusion parameters to depend on the control variable as well as the state variable we have

$$dx = a(x,u,t)dt + b(x,u,t)dz,$$

where dz is the increment of a standard Wiener process. Applying Ito's Lemma to the value function F, we can write

$$F(x + \Delta x, t + \Delta t) = F(x,t) + F_t(x,t)\Delta t + F_x(x,t)\Delta x + \tfrac{1}{2} \left[(\Delta x)^2 F_{xx}(x,t) + \right. \\ \left. + 2\Delta x \Delta t \, F_{xt}(x,t) + (\Delta t)^2 F_{tt}(x,t) \right] + \dots.$$

Substitute dx for the Wiener process above and letting Δt become sufficiently small, so $\Delta t = dt$. Omitting the terms on $F_{xt}(x,t)$ and $F_{tt}(x,t)$ and With $x' = x + dx$ and $t' = t + dt$ we can rewrite the expression to

$$\begin{aligned} F(x',t') = {} & F(x,t) + F_t(x,t)dt + a(x,t)F_x(x,t)dt + b(x,t)F_x(x,t)dz + \\ & + \tfrac{1}{2}\Big[\Big(a^2(x,t)(dt)^2 + 2a(x,t)b(x,t)(dt)^{3/2} + b^2(x,t)dt\Big) F_{xx}(x,t) \\ & +2\left(a(dt)^2 + b(dt)^{3/2}\right) F_{xt}(x,t) + (dt)^2 F_{tt}(x,t)\Big] + o(dt), \end{aligned}$$

where $o(dt)$ represents remainder terms with dt to a power greater than one. Because the terms in $(dt)^2$, $(dt)^{3/2}$ and the remainder $o(dt)$ go to zero faster than dt in the limit we can omit them. Reminding that $E(dz) = 0$, taking expectations we obtain

$$\begin{aligned} E[F(x',t')] &= F(x,t) + F_t(x,t)dt + a(x,t)F_x(x,t)dt + \tfrac{1}{2}b^2(x,t)F_{xx}(x,t)dt \\ &= F(x,t) + \left[F_t(x,t) + a(x,t)F_x(x,t) + \tfrac{1}{2}b^2(x,t)F_{xx}(x,t)\right] dt. \end{aligned}$$

Because $dF = F(x',t') - F(x,t)$ after plugging the derived expectations term into the "return equilibrium" condition above (the Bellmann equation), we have

$$\rho F(x,t) = \max_u \left\{\pi(x,u,t) + F_t(x,t) + a(x,t)F_x(x,t) + \tfrac{1}{2}b^2(x,t)F_{xx}(x,t)\right\}.$$

Special acknowledgement goes to:
Weisstein, Eric W. "Taylor Series." From *MathWorld*–A Wolfram Web Resource.
http://mathworld.wolfram.com/TaylorSeries.html