MM3001

Lecture 6 : More on limits, continuity

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Anteckningstöd

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En av dina Kamerater behöver hjälp: Om det finns någon intresserad att vara anteckningsstödjare gärna kontakt/mejla till stod@math.su.se, så lämnar de mer info.
Det betalas ut ersättning för detta 100 kr/timme exklusive semesterersättning. Det är tiden för föreläsningen, seminariet, lektionen eller liknande som ersätts. Att om studenten ändå antecknar via dator/penna att detta inte innebär något större extra arbete och att det skulle underlätta mycket för en kurskamrat.

Questions?

Lecture Goal and Outcome

Goal: Understand continuous functions and their properties.
Compute limits with l'Hôpital rule.

Learning Outcome: At the end of today lecture you will be able to

- decide if a function is continuous by looking at its law
- Compute the following

$$\lim_{x \to 1} \frac{\ln(x) - x + 1}{x - 1}$$

Why you should care

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- Continuous functions are the backbone of continuous probability.

Example

Remember the Pareto distribution? The probability functions does not jump, it means that the probability changes smoothly as we change the income x.

- Understanding the limit at infinity of a function will give you an idea of its trend. In addition, in many instance compute a probability will amount to compute the limit at $\pm$ infinity of a function.

Example

Since we know that incomes are distributed with the Pareto distribution, what is the mean income that we can expect?

- L'Hôpital rule provides a useful trick to compute limits!

Lecture Plan

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- Continuous functions (7.8)
- Limits at infinty (7.9)
- L'Hôpital rule (7.12)

Section 1
Continuous functions

Continuous function

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Definition

We say that a function f defined on an interval I is continuous at $a \in I$ if the following two conditions hold:

- the limit $\lim_{x \to a} f(x)$ exist.
- we have that $\lim_{x \to a} f(x) = f(a)$.

If f is continuous at every point in I we say that f is continuous on I.

Attention

The point a has to be a point in the domain of the function!

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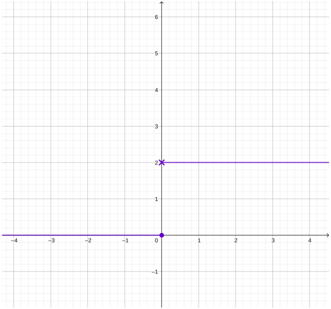
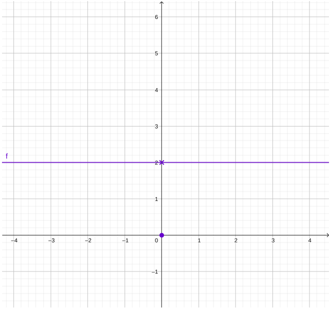
Heuristically this means that you can draw the graph around a without lifting your pencil.

What can go wrong

- The limit does not exist.
- The limit exists, but the function is not defined in the point. The function $f(x) = \frac{e^x - 1}{x}$ is not continuous at 0.
- The limit exists, the function is defined in the point, but its value is not equal to the limit.





Example

Exam question - First Learning outcome

Let a be a real number and let $f(x)$ the following function depending on a

$$f(x) = \begin{cases} \sqrt{ax} & \text{if } x > 2 \\ x - a & \text{if } x \leq 2 \end{cases}$$

Determine for which value of a this is a continuous function.

Properties of continuous functions

Let f and g be two continuous functions on their domain, and let $c \in \mathbb{R}$. Then we have that

- $c \cdot f(x)$, $f(x) + g(x)$, $f(x) - g(x)$, and $f(x)g(x)$ are continuous functions.
- $f(x)/g(x)$ is continuous at every point a such that $g(a) \neq 0$.
- The composition $f \circ g(x)$ is continuous whenever it is defined
- The inverse function $f^{-1}(x)$ is continuous.

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Attention!!!

A function can be continuous at a point, but its derivative can be discontinuous there. Think about $|x|$!

Important

Suppose f and g are two functions. Let a be in the domain of g and suppose that f is a continuous around $g(a)$ then we have that

$$\lim_{x \to a} f \circ g(x) = f\left(\lim_{x \to a} g(x)\right)$$

Questions?

Section 2
More on limits

Infinite limits - Example

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Consider the function

$$f(x) = \frac{1}{|x|}$$

What happens if we get closer and closer to 0?

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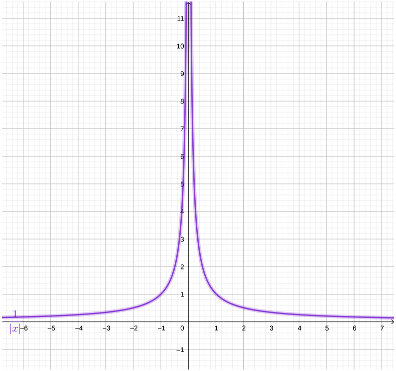

x	$f(x)$
-1	1
-0.1	10
-0.01	100
-0.001	1000
0	?
0.001	1000
0.01	100
0.1	10
1	1

Infinite limits

If, when we get closer and closer to a point a, $f(x)$ gets bigger and bigger (respectively smaller and smaller) then we say that when x tends to a, $f(x)$ tends to $+\infty$ (respectively $-\infty$). We write

$$\lim_{x \to a} f(x) = +\infty \quad \left(\lim_{x \to a} f(x) = -\infty \right)$$

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In this case, we say that the line $x = a$ is a vertical asymptote to the graph of the function. In general the same rule of calculations apply to limits at infinity as we can easily imagine what $a \pm \infty$, $a \times (\pm\infty)$, $\infty \times (\pm\infty)$ is. But we have to be careful.

Indetermined forms

$$0 \cdot (\pm\infty) \quad \frac{\pm\infty}{\pm\infty}, \quad +\infty - \infty, \quad \frac{0}{0}$$

If we get one of these we have to refine our calculations.

Left and right limits - Example

Suppose you throw a tetrahedral dice. Let $f(x)$ denotes the probability that the results is less than, or equal to x. How does $f(x)$ looks like?

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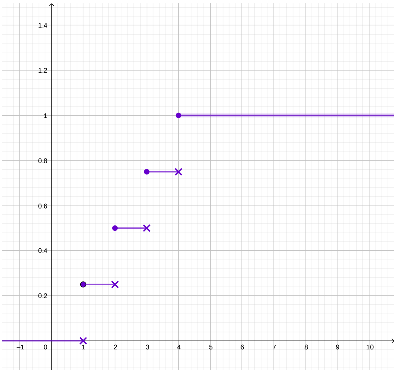

If we approach 1, 2, 3, and 4 on one side, we have that the limit exists. The problem is that it is different depending which side we are approaching from

Left and right limits

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Definition

If $f(x)$ tends to a number B when x approaches to a from smaller values (resp. bigger values) we say that the *B is the limit of $f(x)$ as x tends to a from below (resp. above)* and we write

$$\lim_{x \to a^-} f(x) = B \quad \left(\lim_{x \to a^+} f(x) = B \right)$$

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- The same rules we learned for limits work in this case
- We have that $\lim_{x \to a} f(x)$ exist iff both the limit from above and below exist and they are equal
- One can speak of one sided continuity. One side continuous functions appear EVERYWHERE in probability/time series analysis.

Questions?

Limits at infinity

We can use the language of limits to describe the behaviour of a function when the argument x gets arbitrarily large or small.

Definition

We say that a function $f(x)$ tends to $L \in \mathbb{R} \cup \{\pm\infty\}$ when x goes to $+\infty$ (respectively $-\infty$) if we can make $f(x)$ arbitrarily close to L by sending x to infinity (- infintiy). We write

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On the exam (even under disguise)

Limits at infinity appears on exams. Usually in the "study of function" exercise. But they might also be under disguise in the integral exercise (we are learning integrals in Lecture 8). In order to compute definite integral on infinite intervale one needs to use limits at infinity!

Examples

Compute the following limits

- $\lim_{x\to+\infty} \frac{x^5+x^4+3x+1}{2x^5+2x+1}$
- $\lim_{x\to+\infty} \ln(x+3) - \ln(x)$

Questions?

Section 3
L' Hôpital's rule

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Suppose that f and g are two functions such that

- they are differentiable around a, with the possible exception of a;
- $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

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Attention!

If you want to use L'ôpital rule in the exam you **have to check the assumption to get full points**.

Consequences

- $\lim_{x \to +\infty} \frac{x^n}{e^x} = 0$
- $\lim_{x \to +\infty} \frac{x^n}{\ln x} = +\infty$

Examples

Compute the following limits.

- (From an old exam) $\lim_{x\to 0} \ln(x)x$
- $\lim_{x\to 0} \frac{e^{x^2}-1}{x^2}$
- $\lim_{x\to +\infty} \frac{(2x-1)^4}{(x-1)(x+3)}$
- $\lim_{x\to +\infty} \sqrt{\frac{x^2-e^{-x}+5}{x^2+2x+3}}$
- $\lim_{x\to +\infty} \frac{e^{2x}+x^5-\ln(x)}{\sqrt{1+e^{4x}}}$

Questions?

Thank you for your attention!