Solution to lecture 6 slide

$$f(x) = \begin{cases} \sqrt{ax} & x > 2 \\ x-a & x \leq 2 \end{cases}$$

For which a this function is continuous

First in order to be defined we need

$ax \geq 0$ when $x > 0$ thus $\boxed{a \geq 0}$

For continuity the two branches have to agree in $x=2$ that is

$$\sqrt{a \cdot 2} = 2-a \qquad (*)$$

we square both sides

$$2a = 4 - 4a + a^2$$

$$a^2 - 6a + 4 = 0$$

we use quadratic formula

$$a_{\pm} = 3 \pm \sqrt{9-4} = 3 \pm \sqrt{5}$$

these are both positive but we have to check that they are indeed roots for (*): by squaring we could have produced false roots:

$$\sqrt{2(3+\sqrt{5})} \overset{?}{=} 2-3-\sqrt{5}$$

NO $\sqrt{x} \geq 0$ but the RHS is neg.

$$\underset{1.236068}{\sqrt{6-2\sqrt{5}}} = \underset{1,236067}{2-3+\sqrt{5}}$$

Answer the function is continuous when

$$\boxed{a = 3-\sqrt{5}}$$

- $$\lim_{x \to +\infty} \frac{x^5+x^4+3x+1}{2x^5+2x+1} =$$

$$= \frac{\cancel{x^5}\left(1+\overset{\to 0}{1/x}+\overset{\to 0}{\frac{3}{x^4}}+\overset{\to 0}{1/x^5}\right)}{\cancel{x^5}\left(2+\overset{\to 0}{2/x^4}+\overset{\to 0}{1/x^5}\right)} = \frac{1}{2}$$

- $\lim_{x \to +\infty} \ln(x+3) - \ln(x) = \infty - \infty$ indet

$$= \lim_{x \to +\infty} \ln\left(\frac{x+3}{x}\right) = \ln \text{ cont.}$$

$$= \ln\left(\lim_{x \to +\infty} \frac{x+3}{x}\right)$$

$$= \ln\left(\lim_{x \to +\infty} \frac{\cancel{x}(1+\cancel{3/x}^{\to 0})}{\cancel{x}}\right) = \ln(1) = 0$$

- $\lim_{x \to 0} \ln(x) \cdot x = -\infty \cdot 0$ undef.

$$= \lim_{x \to 0} \frac{\ln(x)}{\frac{1}{x}} \overset{H}{=} \lim_{x \to 0} \frac{\frac{1}{x}}{-1\frac{1}{x^2}} =$$

$$= \lim_{x \to 0} \frac{1}{\cancel{x}} \cdot \frac{-x^{\cancel{2}}}{1} = 0$$

- $$\lim_{x\to 0} \frac{e^x - 1}{x^2} = \frac{0}{0} \text{ undef}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{e^x}{2x} = \begin{cases} \to -\infty & \text{if } x \to 0^- \\ \to +\infty & \text{if } x \to 0^+ \end{cases}$$

The limit is not defined

- $$\lim_{x\to +\infty} \frac{(2x-1)^4}{(x-1)(x+3)} = \lim_{x\to +\infty} \frac{2^4 x^4 + \text{terms of lower deg}}{x^2 + 2x - 3}$$

$$= \lim_{x\to +\infty} \frac{\cancel{x^4}^{\,2}\left(1 + \text{terms that go to } 0\right)}{\cancel{x^2}\left(1 + \cancel{2/x}^{\to 0} - \cancel{3/x^2}^{\to 0}\right)}$$

$$= \lim_{x\to +\infty} x^2 = +\infty$$

- $$\lim_{x\to +\infty} \sqrt{\frac{x^2 - e^{-x} + 5}{x^2 + 2x + 3}} = \sqrt{\ } \text{ cont}$$

$$= \sqrt{\lim_{x\to +\infty} \frac{x^2 - e^{-x} + 5}{x^2 + 2x + 3}} = \sqrt{\frac{\infty}{\infty}}$$

$$= \sqrt{\lim_{x\to +\infty} \frac{\cancel{x^2}\left(1 - \cancel{e^{-x}/x^2}^{\to 0} + \cancel{5/x^2}^{\to 0}\right)}{\cancel{x^2}\left(1 + \cancel{2/x}^{\to 0} + \cancel{3/x^2}^{\to 0}\right)}} = \sqrt{1} = 1$$

$$\lim_{x \to +\infty} \frac{e^{2x} + x^5 - \ln(x)}{\sqrt{1+e^{4x}}} =$$

$$= \lim_{x \to +\infty} \frac{\cancel{e^{2x}}\left(1 + \overset{1}{x^5/e^{2x}} - \overset{2}{\ln(x)/e^{2x}}\right)}{\cancel{e^{2x}}\sqrt{1+\cancel{1/e^{4x}}^{\to 0}}} = (*)$$

Now we check that 1 and 2 goes to zero when $x \to +\infty$

$$\lim_{x \to +\infty} x^5/e^{2x} \overset{H}{=} \lim_{x \to +\infty} \frac{5x^4}{2e^{2x}} = \lim_{x \to +\infty} \frac{20x^3}{4e^{2x}}$$

$$= \lim_{x \to \infty} \frac{60x^2}{8e^{2x}} = \lim_{x \to +\infty} \frac{120x}{16e^{2x}}$$

$$= \lim_{x \to +\infty} \frac{120}{32e^{2x}} = 0$$

$$\lim_{x \to +\infty} \frac{\ln(x)}{e^{2x}} \overset{H}{=} \lim_{x \to +\infty} \frac{1}{2xe^{2x}} = 0$$

Thus

$(*) = 1$