

MM3001

Lecture 10/11: Functions of several variables

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Questions?

Lecture Goal and Outcome

Goals:

- Identify functions of two variables
- Determine the first-order partial derivatives of functions
- Determine second-order partial derivatives,
- Use the chain rule
- Find stationary points for a function.

Learning Outcome: After today you will be able to solve problems like the following:

Problem

Find all the stationary points of the function

$$f(x, y) = (4 - x - y^2)(x + 1).$$

Why you should care

Problem

A firm produces two different kinds of commodities A, B . The daily cost of producing x units of A and y units of B is given by

$$C(x, y) = 4x^2 + xy + y^2 + 400x + 200y + 500,$$

If the firm sells its production at a price per unit 15 for A and 9 for B , how can we find the daily production levels x, y that maximise the profit?

If x and y are related, say $y = y(t)$ and $x = x(t)$, how can we maximise that profit?

Cobb-Douglas model

Productivity depends on several variables such as labor, capital and raw materials. For instance, the Cobb-Douglas model depending only on capital and Labour is given by

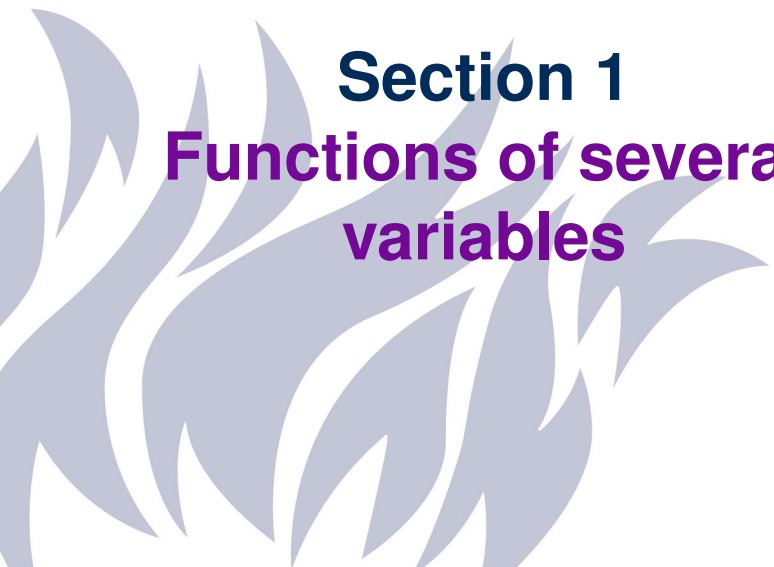
$$Y = f(K, L) := AK^{\alpha}L^{\beta},$$

where

- $Y \equiv$ total production
- $L \equiv$ Labour input
- $K \equiv$ Capital invested
- $A \equiv$ total factor productivity.
- α, β are the output elasticities.

Lecture Plan

- §14.1 Functions of Two Variables
- §14.2 Partial Derivatives with Two Variables
- §14.3 Geometric Representation
- §14.4 Surfaces and Distance
- §14.5 Functions of More Variables
- §15.1 A Simple Chain Rule
- §17.1 Two Variables: Necessary Conditions
- (§17.7 Comparative statistics and the envelope Theorem)



Section 1

Functions of several variables

Functions of two variables

A company produces two goods (A, B) in quantities x, y respectively. The total cost of producing them is given by

$$C(x, y) = 1000 + 20x + 40y,$$

Let $D \subset \mathbb{R}^2$. A function $f : D \rightarrow \mathbb{R}$ is a rule, that for each $(x, y) \in D$ it gives us a value $f(x, y)$. D is called the domain of definition of f .

Examples:

❶ $f(x, y) = x^2 + y^2$

❷ $f(x, y) = x^2 - y^2$

❸ $f(x, y) = x^2 e^{x+y}$

❹ $f(x, y) = 1/(x^2 + 2y^2 + 1)$

The domain of definition of these functions is ...

If $D \subset \mathbb{R}^n$, we say that f is a function of n -variables.

Domain of definition

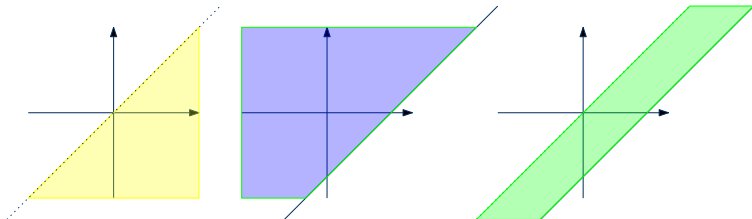
Let $f(x, y) = \ln(x - y) + \sqrt{1 - x + y}$.

The domain of definition of

- $\ln(x - y)$ is all points (x, y) such that $x - y > 0$ (i.e. $y < x$).
- $\sqrt{1 - x + y}$ is all points (x, y) such that $1 - x + y \geq 0$ (i.e. $y \geq x - 1$).

Hence, the domain of definition of f is the set of all points (x, y) satisfying both conditions:

$$\{(x, y) : x - 1 \leq y < x\},$$



Partial derivatives

We define the **partial derivative of f with respect to x** at (x_0, y_0) as

$$\frac{\partial f}{\partial x}(x_0, y_0) = f'_x(x_0, y_0) := \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

We fix y_0 , and differentiate the function as a one-variable function in x

Similarly we define the **partial derivative of f with respect to y** at (x_0, y_0) as

$$\frac{\partial f}{\partial y}(x_0, y_0) = f'_y(x_0, y_0) := \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

Example: If $f(x, y) = x^2 + xy + 3y^2$ then

$$f'_x(x, y) = 2x + y, \quad f'_y(x, y) = x + 6y$$

Examples

We can use all the rules we have learnt to compute partial derivatives:

If $f(x, y) = (x^2 + y)e^{xy}$ then

$$\begin{aligned}f'_x(x, y) &= \left[\frac{\partial}{\partial x}(x^2 + y) \right] e^{xy} + (x^2 + y) \left[\frac{\partial}{\partial x} e^{xy} \right] \\&= [2x] e^{xy} + (x^2 + y) [ye^{xy}] \\&= (2x + x^2y + y^2)e^{xy}\end{aligned}$$

Similarly

$$f'_y(x, y) =$$

Higher order derivatives

We can define

$$f''_{xx}(x, y) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) (x, y) \quad f''_{yx}(x, y) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) (x, y)$$

$$f''_{xy}(x, y) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) (x, y) \quad f''_{yy}(x, y) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) (x, y)$$

Example If $f(x, y) = x^3 e^{y^2}$ then $f'_x(x, y) = 3x^2 e^{y^2}$, $f'_y(x, y) = 2yx^2 e^{y^2}$,

$$f''_{xx}(x, y) = 6x^2 e^{y^2} \quad f''_{yx}(x, y) = 6x^2 y e^{y^2}$$

$$f''_{xy}(x, y) = 6x^2 y e^{y^2} \quad f''_{yy}(x, y) =$$

If the second partial derivatives exists and are continuous then

$$f''_{yx}(x, y) = f''_{xy}(x, y)$$

Geometric representation

A real number x

A pair of numbers (x, y)

A triple of number (x, y, z)

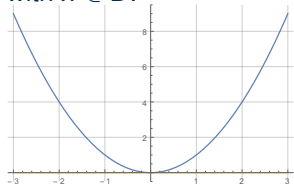
A point on the line $\mathbb{R}$

A point on the plane $\mathbb{R}^2$

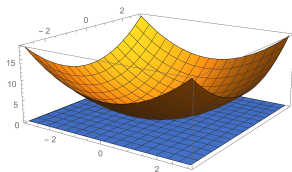
A point in space $\mathbb{R}^3$

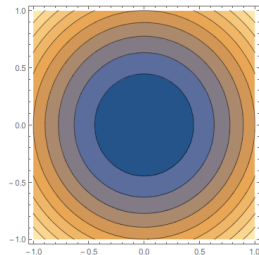
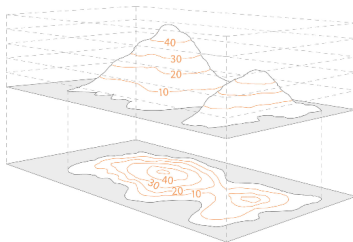
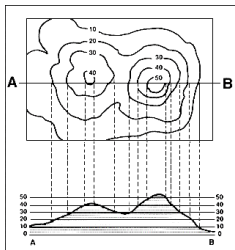
The graph of $f : D \rightarrow \mathbb{R}$ is the set of points

If $D \subset \mathbb{R}$
in $\mathbb{R}^2$ of the form $(x, f(x))$
with $x \in D$.



If $D \subset \mathbb{R}^2$
in $\mathbb{R}^3$ of the form
 $(x, y, f(x, y))$
with $(x, y) \in D$.





A way of representing graphically a function f is by plotting in the (x, y) -plane the sets of points satisfying that

$$f(x, y) = c$$

for a range of constants c . These are called **level curves of f** .

Distance

Given two points in the plane $X = (x, y)$, $Y = (a, b)$ the distance between them is

$$d(X, Y) := \sqrt{|x - a|^2 + |y - b|^2},$$

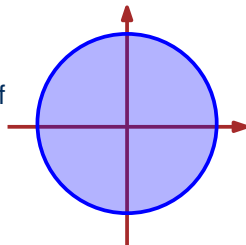
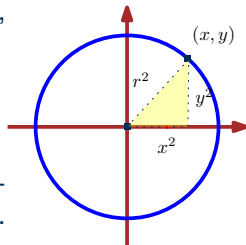
So, if $Y = (0, 0)$, $f(x, y) := \sqrt{x^2 + y^2}$ represents the distance of (x, y) to the origin. The level curves of the form

$$x^2 + y^2 = r^2, \quad r > 0$$

are circle of radius c . Also, the sets of points (x, y) such that

$$x^2 + y^2 \leq r^2, \quad r > 0$$

are disks of radius r .



Given two points in the plane $X = (x, y, z)$, $Y = (a, b, c)$ the distance between them is

$$d(X, Y) := \sqrt{|x - a|^2 + |y - b|^2 + |z - c|^2}$$

So, $\sqrt{x^2 + y^2 + z^2}$ represents the distance of (x, y, z) to the origin.

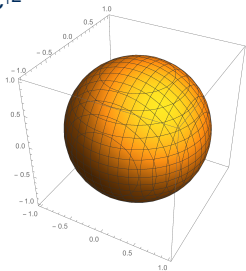
The set of points (x, y, z) such that

$$x^2 + y^2 + z^2 = r^2, \quad r > 0$$

is a sphere of radius r . Also, the set of points (x, y, z) such that

$$x^2 + y^2 + z^2 \leq r^2, \quad r > 0$$

is a (solid) ball of radius r .





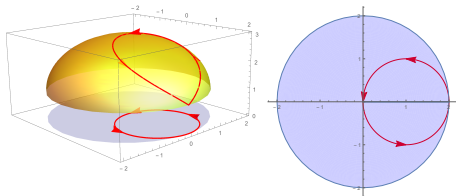
Section 2

The chain rule

Chain rule

Assume that $f(x, y)$ is a function of two variables and both $x = x(t)$, and $y = y(t)$ are functions of one variable t . Then we can consider the one-variable function

$$g(t) := f(x(t), y(t))$$



The points $t \mapsto (x(t), y(t))$ can be thought describing a **curve** on the domain of f . So g describes the values of f along that curve.

The chain rule

$$\frac{d}{dt}g(t) = \frac{\partial}{\partial x}f(x(t), y(t))\frac{d}{dt}x(t) + \frac{\partial}{\partial y}f(x(t), y(t))\frac{d}{dt}y(t)$$

Example

If $f(x, y) = x^2y + 3y$, and $\begin{cases} x(t) = t^2 \\ y(t) = t^3 \end{cases} \quad t \in \mathbb{R}$ then by the chain rule

$$\begin{aligned} \frac{d}{dt}f(x(t), y(t)) &= \frac{\partial}{\partial x}f(x(t), y(t))\frac{d}{dt}x(t) + \frac{\partial}{\partial y}f(x(t), y(t))\frac{d}{dt}y(t) \\ &= \end{aligned}$$

Example: Envelope theorem

Suppose that f depends on a variable x and a parameter t . For a fixed t , let $x^*(t)$ be the value such that

$$f^*(t) := \min(\max)_x f(x, t) = f(x^*(t), t) \quad (\text{the value function}),$$

Assume that such $x^*(t)$ exists, depends smoothly on t , and that it belongs to the interior of the domain of variation of x . In particular

$$\frac{\partial}{\partial x} f(x^*(t), t) = 0 \quad (\text{first-order conditions})$$

The chain rule yields the so-called Envelope Theorem:

$$\frac{d}{dt} f^*(t) = \frac{\partial}{\partial x} f(x^*(t), t) \frac{d}{dt} x^*(t) + \frac{\partial}{\partial y} f(x^*(t), t) \frac{d}{dt} t = \frac{\partial}{\partial y} f(x^*(t), t)$$

$$\partial_{t_j} f^*(t) = \partial_{y_j} f(x^*(t), t) \quad t = (t_1, \dots, t_n) \in \mathbb{R}^n$$

Let $f(x, r_1, r_2) = x^{r_1} - r_2 x$ for $x \geq 0$, where $0 < r_1 < 1$.
This function is concave, and any solution of

$$\max_x f(x, r)$$

is positive, so a solution satisfies the first-order condition

$$r_1 x^{r_1-1} - r_2 = 0 \Rightarrow x^*(r) = (r_1/r_2)^{1/(1-r_1)},$$

So that the value function is

$$f^*(r) = (x^*(r))^{r_1} - r_2 x^*(r) = (r_1/r_2)^{r_1/(1-r_1)} - r_2 (r_1/r_2)^{1/(1-r_1)}$$

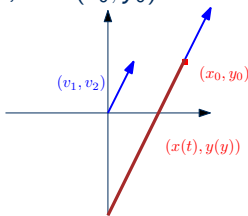
Hence

$$\begin{aligned} \frac{\partial}{\partial r_1} f^*(r) &= \frac{\partial}{\partial r_1} (x^*(r), r) = (x^*(r))^{r_1} \ln(x^*(r)) \\ &= \frac{1}{1-r_1} (r_1/r_2)^{r_1/(1-r_1)} \ln(r_1/r_2) \end{aligned}$$

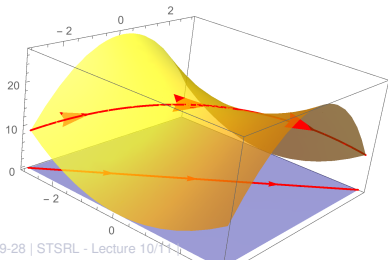
Example: Directional derivatives

Fix (v_1, v_2) a vector in the plane, and $(x_0, y_0) \in \mathbb{R}^2$ and let

$$\begin{cases} x(t) = x_0 + tv_1 \\ y(t) = y_0 + tv_2 \end{cases} \quad t \in \mathbb{R}$$



This describes a line through the point (x_0, y_0) with direction (v_1, v_2) .



So $f(x(t), y(t))$ gives the values of f along that line.

$$\begin{cases} x(t) = x_0 + tv_1 \\ y(t) = y_0 + tv_2 \end{cases} \quad t \in \mathbb{R}. \text{ By the chain rule}$$

$$\begin{aligned} \frac{d}{dt}f(x(t), y(t)) &= \frac{\partial}{\partial x}f(x(t), y(t))v_1 + \frac{\partial}{\partial y}f(x(t), y(t))v_2 \\ &= \begin{bmatrix} \frac{\partial}{\partial x}f(x(t), y(t)) & \frac{\partial}{\partial y}f(x(t), y(t)) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{aligned}$$

For $t = 0$

$$D_v f(x_0, y_0) := \frac{d}{dt}f(x(t), y(t))|_{t=0} = \overbrace{\begin{bmatrix} \frac{\partial}{\partial x}f(x_0, y_0) & \frac{\partial}{\partial y}f(x_0, y_0) \end{bmatrix}}^{Df(x_0, y_0)} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

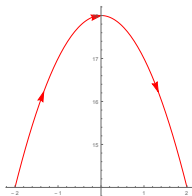
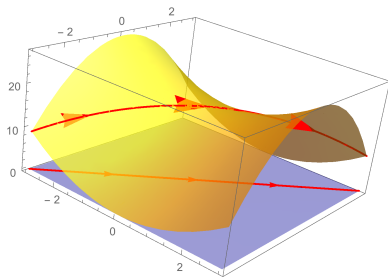
$D_v f(x_0, y_0)$ is called the directional derivative of f in the direction v .

$Df(x_0, y_0)$ is called the total derivative of f at (x_0, y_0) .

Note that $D_{e_1}f(x_0, y_0) = \frac{\partial}{\partial x}f(x_0, y_0)$, $D_{e_2}f(x_0, y_0) = \frac{\partial}{\partial y}f(x_0, y_0)$

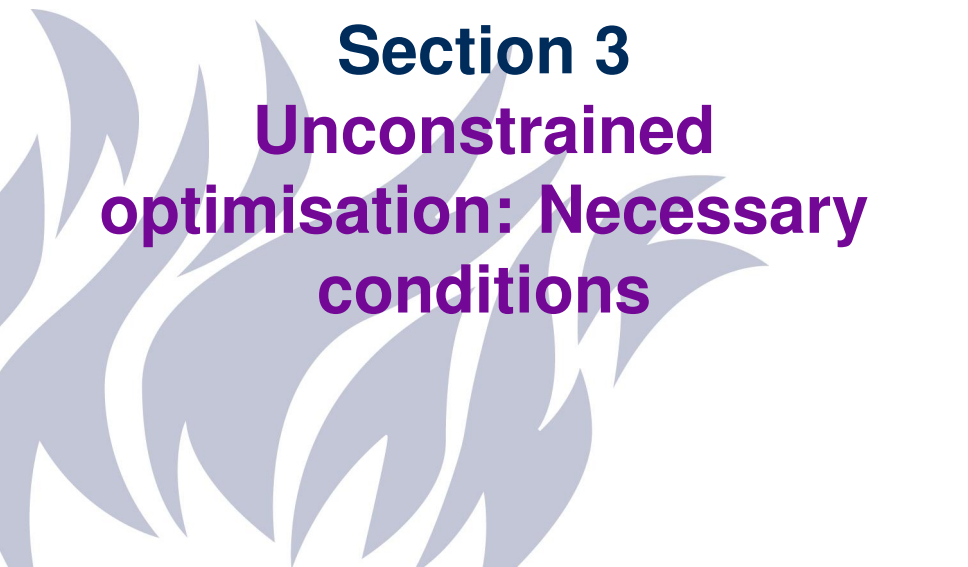
Lets take in particular

$$f(x, y) = 18 + x^2 - 2y^2, \quad \begin{cases} x(t) = t \\ y(t) = t \end{cases}, t \in \mathbb{R} \quad g(t) = f(t, t) = 18 - t^2$$



Then $D_{(1,1)}f(0,0) = g'(0) = 0$.

Note that $D_{e_1}f(0,0) = D_{e_2}f(0,0) = 0$



Section 3

Unconstrained optimisation: Necessary conditions

Stationary points

We say that a point (x_0, y_0) is a **stationary point (critical point)** if

$$\frac{\partial}{\partial x} f(x_0, y_0) = 0 \quad \text{AND} \quad \frac{\partial}{\partial y} f(x_0, y_0) = 0,$$

i.e. $Df(x_0, y_0) = (0, 0)$.

Examples

The point $(0, 0)$ is a stationary point for the functions $f(x, y) = x^2 + y^2$ and $f(x, y) = x^2 - y^2$

Theorem

Given $f : D = [a, b] \rightarrow \mathbb{R}$, if x_0 lies inside the domain of definition D , and x_0 is a local extremum, then $f'(x_0) = 0$. That is, x_0 is a **stationary point (critical point)**.

The set $[a, b]$ is **closed** (i.e. contain its boundary) and **bounded** (i.e. we can enclose it within a circle).

Sets that are closed and bounded are called **compact**.

Theorem

Let $D \subset \mathbb{R}^2$ be a compact set. Given $f : D \rightarrow \mathbb{R}$, if (x_0, y_0) lies inside the domain of definition D , and (x_0, y_0) is a local extreme, then $Df(x_0, y_0) = (0, 0)$. That is, (x_0, y_0) is a **stationary point (critical point)**.

Example

Let $f(x, y) = x^2 + y^2$ on the set of points D with $x^2 + y^2 \leq 1$.
The set D is a disk, centered at $(0, 0)$ of radius 1.

Note that

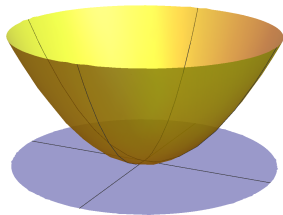
$$\begin{cases} \frac{\partial}{\partial x} f(x, y) = 2x = 0 \Leftrightarrow x = 0 \\ \frac{\partial}{\partial y} f(x, y) = 2y = 0 \Leftrightarrow y = 0 \end{cases}$$

So $(0, 0)$ is the only stationary point.

Clearly for all (x, y) ,

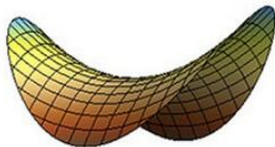
$$f(x, y) = x^2 + y^2 \geq 0 = f(0, 0)$$

so $(0, 0)$ is a (local) minimum for f .



Not every stationary point is a local extreme.

Example The function $f(x, y) = x^2 - y^2$ defined on the set of points $x^2 + y^2 \leq 1$, satisfies that $(0, 0)$ is a stationary point inside the domain, but it is not an extreme point.



$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$$



Thank you for your attention!

