Motivating example: Linear Convolutional Networks (LCNs)

$$\varphi_{LCN}: \mathbb{R}^2 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^5$$

$$(w, v) \longmapsto \underbrace{\begin{bmatrix} w_0 & w_1 \end{bmatrix}}_{\text{layer 2}} \cdot \underbrace{\begin{bmatrix} v_0 & v_1 & v_2 & & \\ & & v_0 & v_1 & v_2 \end{bmatrix}}_{\text{layer 1}}$$

How many linear maps does this 2-layer network parametrize?

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5 parameters: w_0, w_1, v_0, v_1, v_2

$$\dim(\mathrm{im}(\varphi_{LCN})) = 4$$

This dimension measures the exact expressivity of the network.

$$\mathrm{im}(\varphi_{LCN}) = \{(a,b,c,d,e) \mid ad^2 + b^2e = bcd \ \& \ c^2 \geq 4ae\}$$

How to compute the dimension of the image of $\varphi: X \to Y$ in general?

① **Directly study $\mathrm{im}(\varphi)$.**

$$\left.\begin{array}{l} \text{E.g., for } \varphi_{LU}: (L, U) \longmapsto L \cdot U \in \mathbb{R}^{n\times n}, \\ \text{show that almost every } A \in \mathbb{R}^{n \times n} \text{ has a LU decomposition} \end{array}\right\} \dim(\mathrm{im}(\varphi_{LU})) = n^2$$

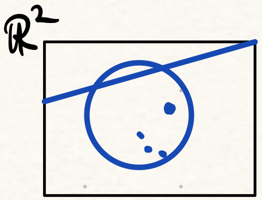
② **Fiber-dimension theorem**

③ **Jacobian check**

What is the dimension of a (semi)algebraic set?

$\boxed{\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}}$

A variety $X \subseteq \mathbb{K}^n$ is **irreducible** if it is not the union of 2 proper subvarieties,
i.e., there are no varieties $X_1, X_2 \subseteq \mathbb{K}^n$ s.t. $X = X_1 \cup X_2$
& $\emptyset \neq X_i \subsetneq X$

Ex.: 

is reducible into 6 irreducible components

Fact: There is a subvariety $\Delta \subsetneq X$ such that $X \setminus \Delta$ is a smooth manifold of dimension k.
If X is irreducible, k is the same for all such Δ: k is the **dimension** of X.

$$\dim(\bigcirc) = 1, \quad \dim(\cdot) = 0$$

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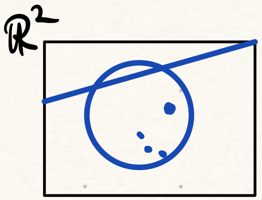
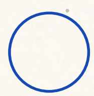
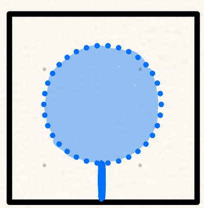
Ex.: $\mathbb{R}^2$  is reducible into 6 irreducible components

Fact: There is a subvariety $\Delta \subsetneq X$ such that $\overbrace{X \setminus \Delta}^{\text{Zariski open in } X}$ is a smooth manifold of dimension k. If X is irreducible, k is the same for all such Δ: k is the **dimension** of X.

dim () = 1, $\dim(\cdot) = 0$

In general: $X = \bigcup_i X_i$ ← irreducible varieties
$\rightarrow \dim(X) := \max_i \dim(X_i)$

For a $\left\{\begin{array}{l}\text{semialgebraic set } X \subseteq \mathbb{R}^n \\ \text{constructible set } X \subseteq \mathbb{K}^n\end{array}\right\}$, define $\dim(X) := \dim(\overline{X})$. ← Zariski closure

dim  = dim  = 2

$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

Dimension of Image

Let $\varphi : X \longrightarrow \mathbb{K}^m$ morphism between irreducible varieties.

Zariski closure

Exercise: $Y := \overline{\varphi(X)}$ is irreducible

$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

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Let us first consider the case $X = \mathbb{K}^n$.

It is sufficient to compute the image dimension over $\mathbb{C}$:

If $\mathbb{K} = \mathbb{R}$, φ is a polynomial map with real coefficients.

Exercise: $\dim_{\mathbb{R}} \underset{\mathbb{R}^m}{\underset{\cap}{\varphi(\mathbb{R}^n)}} = \dim_{\mathbb{C}} \underset{\mathbb{C}^m}{\underset{\cap}{\varphi(\mathbb{C}^n)}}$

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Exercise: $\dim_{\mathbb{R}} \varphi(\mathbb{R}^n) = \dim_{\mathbb{C}} \varphi(\mathbb{C}^n)$

$\varphi(\mathbb{R}^n) \subseteq \mathbb{R}^m$, $\varphi(\mathbb{C}^n) \subseteq \mathbb{C}^m$

Important Principle: Complex AG often easier than real AG

AG = algebraic geometry

Solve real problem over $\mathbb{C}$ first!

Fiber-Dimension theorem: Let $\varphi: \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial map, $Y := \overline{\operatorname{im}(\varphi)}$.

a) $\dim Y \leq n$

b) For all $y \in \operatorname{im}(\varphi)$, every irreducible component of the fiber $\varphi^{-1}(y)$ has dimension $\geq n - \dim Y$.

c) There is a subvariety $\Delta \subsetneq Y$ s.t. $\dim(\varphi^{-1}(y)) = n - \dim Y$ for all $y \in \operatorname{im}(\varphi) \setminus \Delta$.

We can compute $\dim(Y)$ by computing the dimension of a generic fiber!

Do you know this from linear algebra?

Linear Algebra

$\varphi: \mathbb{C}^n \to \mathbb{C}^m$ linear map

$Y := \mathrm{im}(\varphi)$ vector space

every fiber over Y has same dimension

Nonlinear Algebra

$\varphi: \mathbb{C}^n \to \mathbb{C}^m$ polynomial map

$Y := \overline{\mathrm{im}(\varphi)}$ variety

fibers over Y vary in dimension, but almost all fibers have same dimension

Example

$$\varphi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
$$(x,y) \longmapsto (x, xy)$$

Compute $\varphi^{-1}(a,b)$:

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Compute $\varphi^{-1}(a,b)$:

① $a \neq 0$: $\varphi^{-1}(a,b) = \{(a, \frac{b}{a})\}$ 1 point

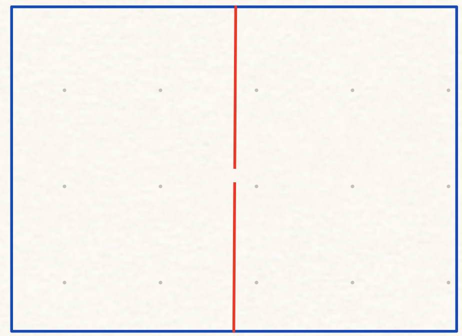
② $a = 0,\ b \neq 0$: $\varphi^{-1}(0,b) = \emptyset$

③ $a = 0,\ b = 0$: $\varphi^{-1}(0,0) = \{(0,y) \mid y \in \mathbb{C}\}$ 1 line

Example

$\mathrm{im}(\varphi) = (\mathbb{C}^2 \setminus L) \cup \{(0,0)\}$

$L = \{0\} \times \mathbb{C}$ line in $\mathbb{C}^2$



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Almost every fiber is a single point.

Linear Algebra

$\varphi : \mathbb{C}^n \to \mathbb{C}^m$ linear map

$Y := \mathrm{im}(\varphi)$ vector space

every fiber over Y has same dimension

rank-nullity theorem:
$n = \dim \ker(\varphi) + \dim Y$

Nonlinear Algebra

$\varphi : \mathbb{C}^n \to \mathbb{C}^m$ polynomial map

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$\boxed{\begin{array}{l}\text{rank-nullity theorem:} \\ n = \dim \ker(\varphi) + \dim Y\end{array}}$

works over $\mathbb{R}$ as well

Nonlinear Algebra

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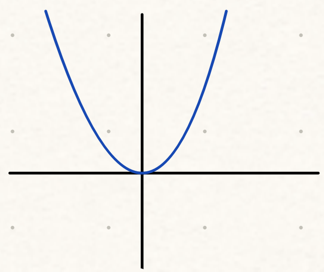
$\boxed{\begin{array}{l}\text{fiber-dimension theorem:} \\ n = \dim \varphi^{-1}(y) + \dim Y \\ \text{for almost all } y \in Y\end{array}}$

does not work over $\mathbb{R}$

Example

$$\varphi : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2$$

$$\operatorname{im}(\varphi) = \mathbb{R}_{\geq 0}$$



$y > 0 \Rightarrow \varphi^{-1}(y) = 2$ points

$y = 0 \Rightarrow \varphi^{-1}(y) = 1$ point

$y < 0 \Rightarrow \varphi^{-1}(y) = \emptyset$

$$\varphi : \mathbb{C} \to \mathbb{C}$$
$$x \mapsto x^2$$

$$\operatorname{im}(\varphi) = \mathbb{C}$$

$\underbrace{\text{For almost all } y \in \mathbb{C}}_{\forall y \neq 0}$, $\varphi^{-1}(y) = 2$ points.

What is the generic fiber of

In other words: Which network parameters give the same function?

$$\varphi_{LCN}: \mathbb{C}^2 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^5$$

$$(w, v) \longmapsto \underbrace{\begin{bmatrix} w_0 & w_1 \end{bmatrix}}_{\text{layer 2}} \cdot \underbrace{\begin{bmatrix} v_0 & v_1 & v_2 & & \\ & & v_0 & v_1 & v_2 \end{bmatrix}}_{\text{la er 1}} = \begin{bmatrix} w_0v_0 & w_0v_1 & w_0v_2 + w_1v_0 & w_1v_1 & w_1v_2 \end{bmatrix}$$

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A quartic polynomial has 4 complex roots & there are finitely many ways, up to scaling, to combine those roots to the 2 layer factors.

(a_1+b_1x) (a_2+b_2x) (a_3+b_3x) (a_4+b_4x)

$w_0+w_1x^2$ $\quad$ $v_0+v_1x+v_2x^2$

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1-dimensional fibers over image

Each of these ways has a 1-dimensional scaling ambiguity: $\lambda(w_0+w_1x^2)\cdot\frac{1}{\lambda}(v_0+v_1x+v_2x^2)$

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$$\xrightarrow[\text{dim. thm.}]{\text{fiber-}} \dim(\operatorname{im}(\varphi_{LCN})) = \dim(\mathbb{C}^2\times\mathbb{C}^3) - 1 = 4$$

Jacobian check: Let $\varphi : \mathbb{K}^n \to \mathbb{K}^m$ be a polynomial map, $Y := \overline{\mathrm{im}(\varphi)}$.

There is a subvariety $\Delta \subsetneq \mathbb{K}^n$ s.t. for all $p \in \mathbb{K}^n \setminus \Delta$,
the rank of the Jacobi matrix $\mathcal{J}_\varphi(p)$ equals $\dim Y$.

$$\begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \varphi_m}{\partial x_1} & \frac{\partial \varphi_m}{\partial x_2} & \cdots & \frac{\partial \varphi_m}{\partial x_m} \end{bmatrix}(p)$$

Practical test: ① Choose a random point $p \in \mathbb{K}^n$
② compute $\mathrm{rank}(\mathcal{J}_\varphi(p))$
③ repeat until confident

works over $\mathbb{R}$ & $\mathbb{C}$

Let $\varphi: \mathbb{K}^n \to \mathbb{K}^m$ be a polynomial map, $Y := \overline{\mathrm{im}(\varphi)}$.

Exercise: $\mathrm{rank}(J_\varphi(p)) \leq \dim Y$ for <u>all</u> $p \in \mathbb{K}^n$

Corollary: Finding a <u>single</u> point $p \in \mathbb{K}^n$ with $\mathrm{rank}(J_\varphi(p)) = m$ is enough to show that $Y = \mathbb{K}^m$ (e.g., for LU decomposition).

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Jacobi matrix: $\quad J_{w,v} = \begin{bmatrix} v_0 & 0 & w_0 & 0 & 0 \\ v_1 & 0 & 0 & w_0 & 0 \\ v_2 & v_0 & w_1 & 0 & w_0 \\ 0 & v_1 & 0 & w_1 & 0 \\ 0 & v_2 & 0 & 0 & w_1 \end{bmatrix}$

Exercise: a) Check that $\operatorname{rank}(J_{w,v}) \leq 4$ for all w, v.

b) For which w and v do we have $\operatorname{rank}(J_{w,v}) \leq 3$?

What about $\operatorname{rank}(J_{w,v}) \leq 2$ and $\operatorname{rank}(J_{w,v}) \leq 1$?

What about morphisms $\varphi : X \to \mathbb{K}^m$?

irreducible variety

Ex: $\mathcal{M}_1^{2\times 2} \longrightarrow \mathbb{K}^3$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} & b \\ c & d \end{bmatrix}$

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For a fiber-dimension theorem to hold, we need to work again over $\mathbb{C}$.

If $\mathbb{K} = \mathbb{R}$ and $X \subseteq \mathbb{R}^n$, we consider

$X_{\mathbb{C}}$:= Zariski closure of X inside $\mathbb{C}^n$ (i.e., the smallest variety in $\mathbb{C}^n$ containing X)

Fact: $\dim_{\mathbb{R}} \varphi(X) = \dim_{\mathbb{C}} \varphi(X_{\mathbb{C}})$

Fiber-Dimension theorem: Let $\varphi: X \to \mathbb{C}^m$ be a polynomial map, $X \subseteq \mathbb{C}^n$ irreducible variety, $Y := \overline{\varphi(X)}$.

a) $\dim Y \leq \dim X$

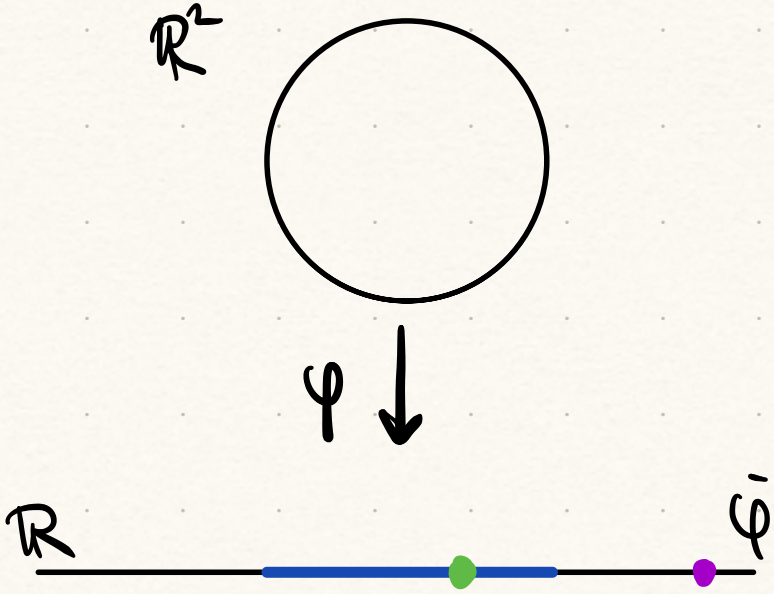
b) For all $y \in \mathrm{im}(\varphi)$, every irreducible component of the fiber $\varphi^{-1}(y)$ has dimension $\geq \dim X - \dim Y$.

c) There is a subvariety $\Delta \subsetneq Y$ s.t. $\dim(\varphi^{-1}(y)) = \dim X - \dim Y$ for all $y \in \mathrm{im}(\varphi) \setminus \Delta$.

$$\boxed{\text{This theorem also holds for irreducible projective varieties } X \subseteq \mathbb{P}^n_{\mathbb{C}} \text{ and homogeneous polynomial maps } \varphi: X \to \mathbb{P}^m_{\mathbb{C}}.}$$

Example: $\varphi : \{(x,y) \in \mathbb{K}^2 \mid x^2 + y^2 - 1 = 0\} \longrightarrow \mathbb{K}$

$$(x,y) \longmapsto x$$



$\varphi^{-1}(\bullet) = 2$ points

$$\varphi^{-1}(\bullet) = \begin{cases} \varnothing & \text{if } \mathbb{K} = \mathbb{R} \\ 2 \text{ points} & \text{if } \mathbb{K} = \mathbb{C} \end{cases}$$

In fact, for $\mathbb{K} = \mathbb{C}$:

$$\varphi^{-1}(x) = \begin{cases} 2 \text{ points} & \text{if } x \neq \pm 1 \\ 1 \text{ point} & \text{if } x = 1 \text{ or } -1 \end{cases}$$

Example: $\varphi : \{(x:y:z) \in \mathbb{P}^2_{\mathbb{K}} \mid x^2+y^2-z^2=0\} \longrightarrow \mathbb{P}^1_{\mathbb{K}}$

$(x:y:z) \longmapsto (x:z)$

one-dimensional curve in projective plane

one-dim. projective space

Exercise: For $\mathbb{K}=\mathbb{C}$: $\varphi^{-1}(1:1) = \{(1:0:1)\}$,

$\varphi^{-1}(1:-1) = \{(1:0:-1)\}$,

$\varphi^{-1}(x:z) =$ 2 points if $(x:z) \notin \{(1:1),(1:-1)\}$

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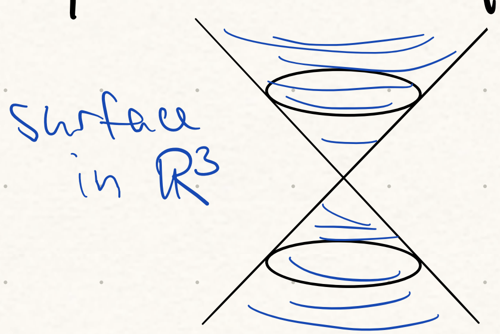
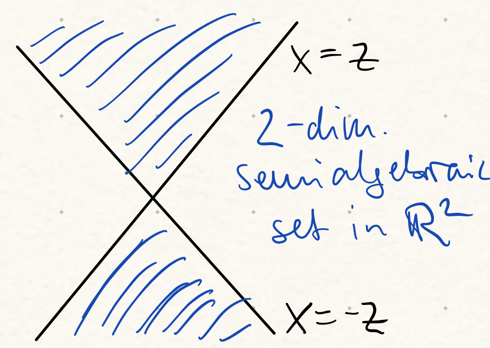
Also when working with affine cones, the generic complex fiber is 2 points:

$\{(x,y,z) \in \mathbb{K}^3 \mid x^2+y^2-z^2=0\} \to \mathbb{K}^2$

$(x,y,z) \mapsto (x,z)$

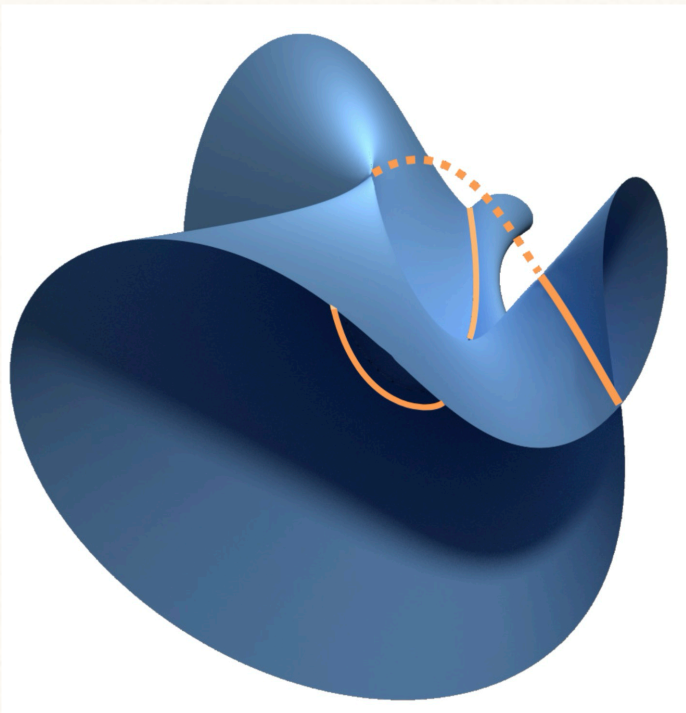
$\mathbb{K}=\mathbb{R}$:

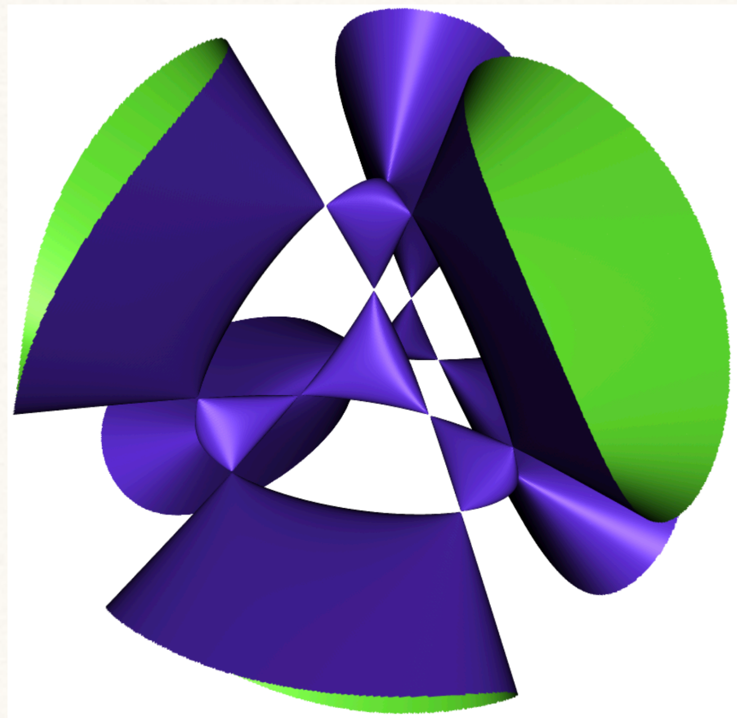




For a Jacobian check, we need to restrict the Jacobi matrices to the tangent spaces of X.

Issue: In general, varieties have singularities.





Singularities & Tangent Spaces

- $X \subseteq \mathbb{K}^n$ irreducible variety

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- $I(X) := \{ f \in \mathbb{K}[x_1, \dots, x_n] \mid \forall p \in X : f(p) = 0 \}$ vanishing ideal

Singularities & Tangent Spaces

- $X \subseteq \mathbb{K}^n$ irreducible variety
- $I(X) := \{ f \in \mathbb{K}[x_1, \ldots, x_n] \mid \forall p \in X: f(p) = 0 \}$ vanishing ideal
- Fix a generating set $\mathcal{F} = \{ f_1, \ldots, f_s \}$ such that $\langle f_1, \ldots, f_s \rangle = I(X)$

  finite by Hilbert's basis theorem

Singularities & Tangent Spaces

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- Fix a generating set $\mathcal{F} = \{ f_1, \ldots, f_s \}$ such that $\langle f_1, \ldots, f_s \rangle = I(X)$

  finite by Hilbert's basis theorem

- Compute jacobian matrix $J_{\mathcal{F}} := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_s}{\partial x_1} & \cdots & \frac{\partial f_s}{\partial x_n} \end{bmatrix}$

Singularities & Tangent Spaces

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can be taken as a definition of $\dim(X)$

Singularities & Tangent Spaces

Definition: Points $p \in X$ with $\operatorname{rank} J_{\mathcal{F}}(p) = n - \dim(X)$ are called regular or smooth.
All other $p \in X$ are called singular.

Fact: At singular points $p \in X$: $\operatorname{rank} J_{\mathcal{F}}(p) < \underbrace{n - \dim(X)}_{=: c}$.

Corollary: The set of singular points on X is a proper subvariety defined by the polynomials in $I(X)$ and the $c \times c$ minors of $J_{\mathcal{F}}$.

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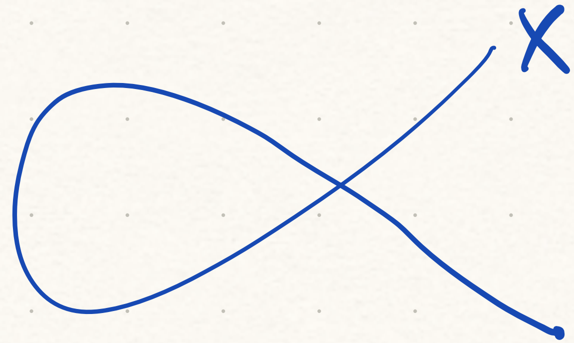
Definition: The tangent space at a smooth point $p \in X$ is

$$T_pX := \underbrace{\ker J_{\mathcal{F}}(p)}_{\text{independent of choice of } \mathcal{F}} \subseteq \mathbb{K}^n$$

$\dim(T_pX) = \dim(X)$

Example

$$y^2 - x^2(x+1) = 0$$



$$I(X) = \langle y^2 - x^2(x+1) \rangle$$

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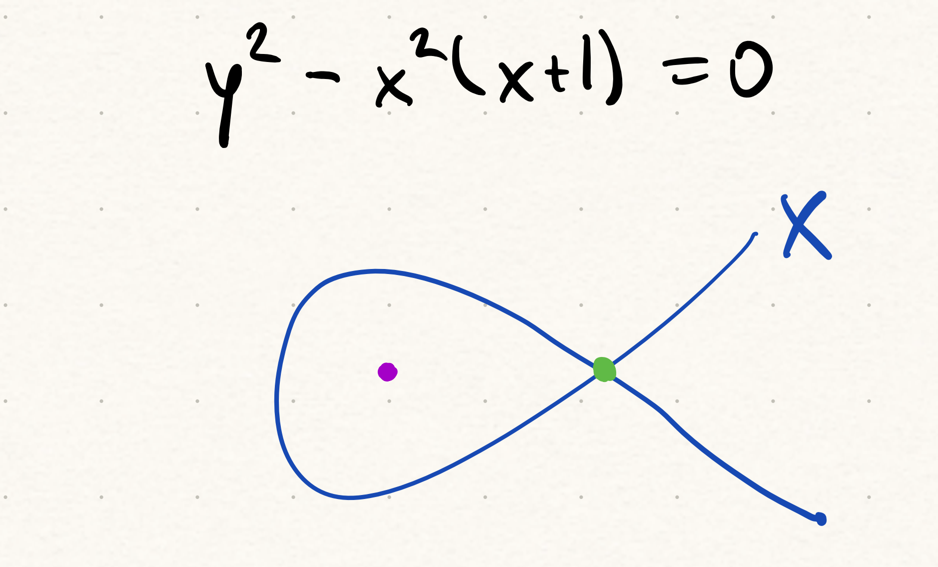

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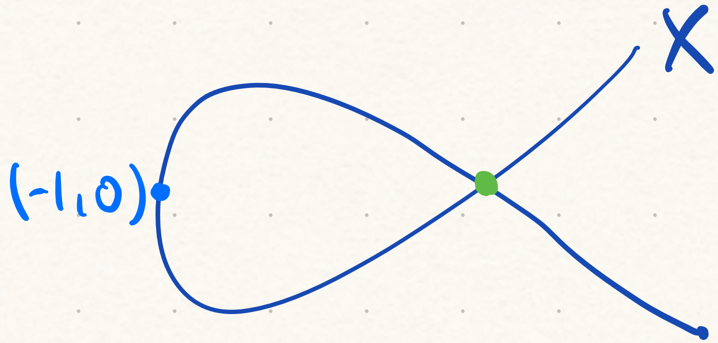
$$\operatorname{rank}(J) = 0 \iff (x,y) \in \{ \underbrace{(0,0)}_{\text{lies on } X},\ \underbrace{(-\tfrac{2}{3},0)}_{\text{does not lie on } X} \}$$

The only singularity of the curve X is the origin $(0,0)$.

Example

$$y^2 - x^2(x+1) = 0$$



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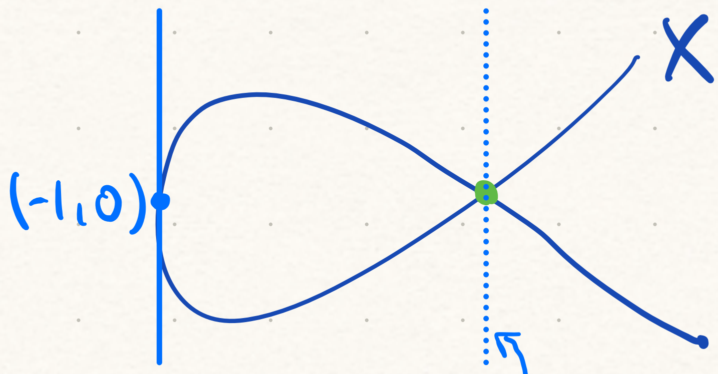

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$$T_{(-1,0)} X = \ker J((-1,0)) = \{(0,y) \mid y \in \mathbb{K}\}$$

Jacobian check: Let $\varphi : X \to \mathbb{K}^m$ be a polynomial map, $X \subseteq \mathbb{K}^n$ irreducible variety, $Y := \overline{\varphi(X)}$.

- For $p \in X$, $J_\varphi(p)$ defines a linear map $\mathbb{K}^n \to \mathbb{K}^m$.

$$\begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \varphi_m}{\partial x_1} & \frac{\partial \varphi_m}{\partial x_2} & \cdots & \frac{\partial \varphi_m}{\partial x_m} \end{bmatrix}(p)$$

- If p is a smooth point, the restriction of $J_\varphi(p)$ onto the tangent space T_pX is the first-order approximation of φ around p.

There is a subvariety $\Delta \subsetneq X$ s.t. for all $p \in X \setminus \Delta$,
p is a smooth point of X &
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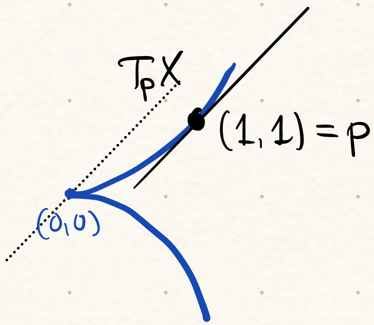
There is a subvariety $\Delta \subsetneq X$ s.t. for all $p \in X \setminus \Delta$, p is a smooth point of X & the rank of $\underbrace{J\varphi(p)\big|_{T_pX}}_{=:\, d_p\varphi}$ equals $\dim Y$.

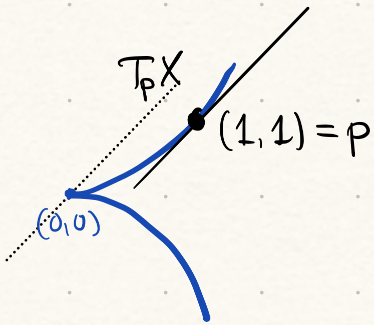
The image of the linear map $d_p\varphi$ is $T_{\varphi(p)}Y$ if $\varphi(p)$ is a smooth point of Y.

Example



$$X = \{(x,y) \mid y^2 - x^3 = 0\}$$

Example



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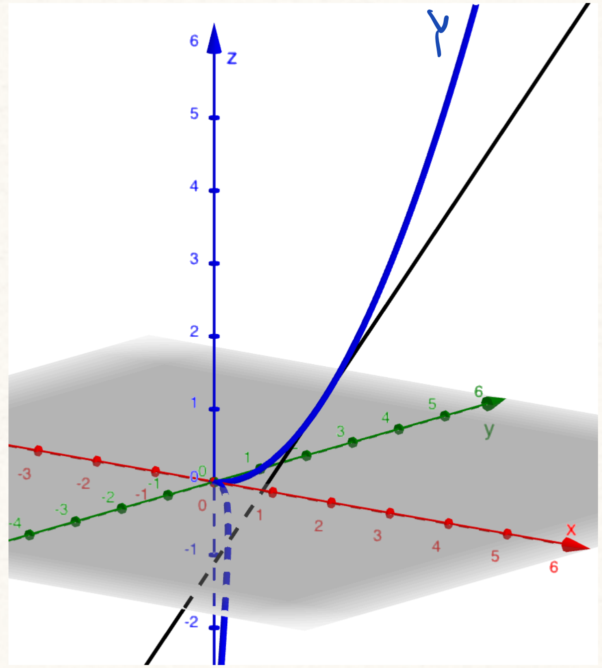
$$\xrightarrow{\quad\varphi\quad}$$

$$(x,y) \longmapsto (x,y,xy)$$



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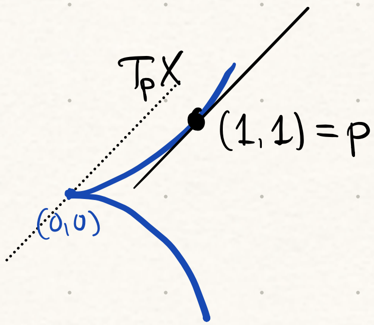

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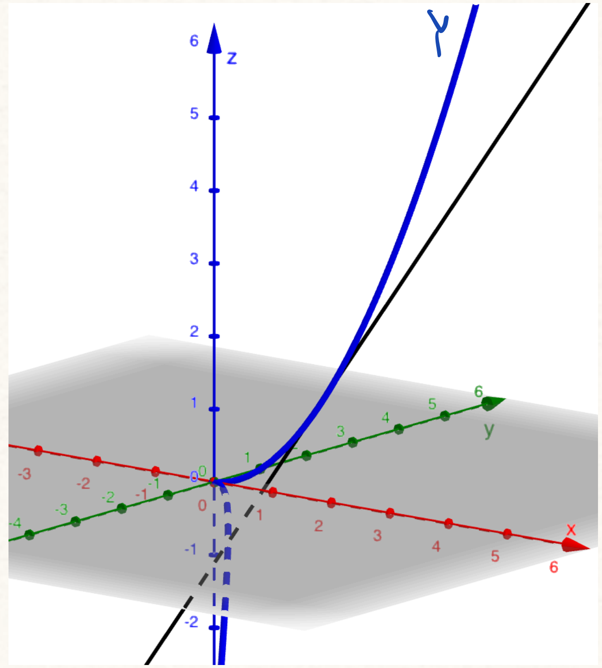


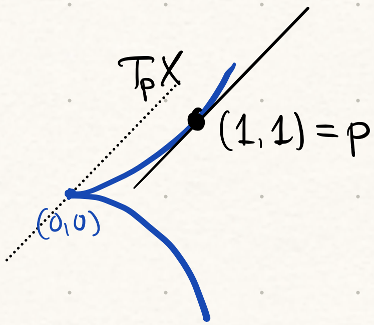
$$J_f = \begin{bmatrix} -3x^2 & 2y \end{bmatrix}$$

$$J_f(p) = \begin{bmatrix} -3 & 2 \end{bmatrix}$$

$$T_pX = \ker J_f(p)$$

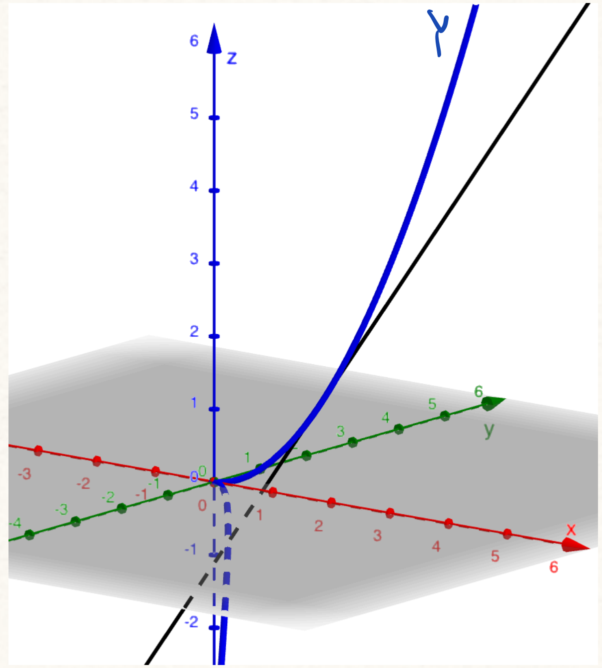
$$= \{(2t, 3t) \mid t \in K\}$$

Example



$$\mathbb{R}^2 \xrightarrow{\varphi} \mathbb{R}^3$$

$$(x,y) \longmapsto (x,y,xy)$$



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$$J_\varphi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y & x \end{bmatrix}$$

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$J_\varphi(p) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ has rank 2

but restricted to T_pX it has rank 1 with image $T_{(1,1,1)}Y = \{(2t,3t,5t) \mid t \in \mathbb{K}\}$

How to compute the dimension of the image of $\varphi: X \to Y$ in general?

① Directly study $\mathrm{im}(\varphi)$.

$$\left.\begin{array}{l} \text{E.g., for } \varphi_{LU}: \left(L, U\right) \longmapsto L \cdot U \in \mathbb{R}^{n\times n}, \\ \quad \text{show that almost every } A \in \mathbb{R}^{n\times n} \text{ has a LU decomposition} \end{array}\right\} \dim(\mathrm{im}(\varphi_{LU})) = n^2$$

② Fiber-dimension theorem

③ Jacobian check