Week 3 – Exercises

September 16, 2026

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Given a nonzero vector $(p_0, p_1, \ldots, p_n) \in \mathbb{K}^{n+1}$, we write $[p_0 : p_1 : \ldots : p_n] \in \mathbb{P}^n_{\mathbb{K}}$ for the corresponding point in projective space.

1 Fibers

Exercise 1. *Consider the map*

$$\varphi : \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} \mid x^2 + y^2 - z^2 = 0\} \longrightarrow \mathbb{P}^1_{\mathbb{C}}, \quad [x : y : z] \longmapsto [x : z].$$

Show that

$$\begin{aligned} \varphi^{-1}([1 : 1]) &= \{[1 : 0 : 1]\}, \\ \varphi^{-1}([1 : -1]) &= \{[1 : 0 : -1]\}, \\ \varphi^{-1}([x : z]) &= \textit{2 points if } [x : z] \notin \{[1 : 1],\ [1 : -1]\}. \end{aligned}$$

Exercise 2. *Consider the map*

$$\begin{aligned} &\tilde{\varphi}_{\mathrm{LCN}} : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \longrightarrow \mathbb{P}^4_{\mathbb{C}}, \\ &([u : v], [x : y : z]) \longmapsto \begin{bmatrix} u & v \end{bmatrix} \cdot \begin{bmatrix} x & y & z & 0 & 0 \\ 0 & 0 & x & y & z \end{bmatrix} = [ux : uy : uz + vx : vy : vz]. \end{aligned}$$

a) Compute the fiber over $\tilde{\varphi}_{\mathrm{LCN}}([1 : 3], [1 : 2 : 5])$.
(You'll see that it consists only of the point $([1 : 3], [1 : 2 : 5])$.)

b) Can you find a point $([u : v], [x : y : z])$ such that the fiber over $\tilde{\varphi}_{\mathrm{LCN}}(([u : v], [x : y : z]))$ consists of two points?

c) More difficult: *Show that all non-empty fibers are finite.*

d) Deduce from c) that the dimension of the image has dimension 3.

e) Challenge: *Can you determine all points $([u : v], [x : y : z])$ such that the fiber over $\tilde{\varphi}_{\mathrm{LCN}}(([u : v], [x : y : z]))$ is a single point?*

Exercise 3. *Consider the map*

$$\begin{aligned} &\varphi_{\mathrm{LCN}} : \mathbb{C}^2 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^5, \\ &((u, v), (x, y, z)) \longmapsto \begin{bmatrix} u & v \end{bmatrix} \cdot \begin{bmatrix} x & y & z & 0 & 0 \\ 0 & 0 & x & y & z \end{bmatrix} = (ux, uy, uz + vx, vy, vz). \end{aligned}$$

a) *Compute the fiber over* $\varphi_{\mathrm{LCN}}((1,3),(2,3,5))$. *Show that it is irreducible and of dimension one.*

b) *Deduce from Exercise 2c) that all non-empty fibers are one-dimensional and that the image of* φ_{LCN} *has dimension 4.*

c) *Check computationally that the rank of the Jacobi matrix of* $\varphi_{\mathbf{LCN}}$ *at all points* $((u,v),(x,y,z))$ *is at most 4.*

d) *For which points* $((u,v),(x,y,z))$ *is the rank of the Jacobi matrix at most 3? What about rank at most 2 or 1? (see also Exercise 7)*

2 Singularities

Exercise 4. *Plot the surface* $\{(x,y,z) \in \mathbb{R}^3 \mid x^2z + y^2 = 0\}$ *and compute its singular locus.*

3 Jacobian check

Exercise 5. *Consider the following* shallow cubic neural network*:*

$$\begin{bmatrix} e & f \end{bmatrix} \sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right), \qquad \textit{where } \sigma\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) := \begin{bmatrix} u^3 \\ v^3 \end{bmatrix},$$

x, y *are inputs to the network, and* $a, b, \ldots, f$ *are network parameters. Note that the left-hand side above equals*

$$(a^3e + c^3f)x^3 + 3(a^2be + c^2df)x^2y + 3(ab^2e + cd^2f)xy^2 + (b^3e + d^3f)y^3.$$

Show that, when varying the network parameters, the network parametrizes a 4-dimensional family of functions in x *and* y*.*
(Hint: Jacobian check; see also Exercise 9a)

Exercise 6. *Let* X *be an irreducible variety and* $\varphi : X \to \mathbb{K}^m$ *be a morphism. Show that the Zariski closure of the image* $\varphi(X)$ *is irreducible.*

Exercise 7. *Show that an* $m \times n$ *matrix is of rank at most* r *if and only if all its* $(r+1) \times (r+1)$ *minors are zero!*
This shows that

$$\mathcal{M}_r^{m\times n} := \{A \in \mathbb{K}^{m\times n} \mid \mathrm{rank}(A) \leq r\}$$

is an algebraic variety, with the $(r+1) \times (r+1)$ *minors being its defining polynomial equations.*

Exercise 8. *Let* $f \in \mathbb{K}[x_1, \ldots, x_n]$*. Prove the following:*

a) *If there is a Euclidean open ball $\emptyset \neq \mathcal{B} \subseteq \mathbb{K}^n$ such that $f(p) = 0$ for all $p \in \mathcal{B}$, then $f = 0$.*
(You may use the following fact: A nonzero univariate polynomial of degree d has at most d roots.)

b) *If there is a proper subvariety $\Delta \subsetneq \mathbb{K}^n$ such that $f(p) = 0$ for all $p \in \mathbb{K}^n \setminus \Delta$, then $f = 0$.*
(Hint: this can be deduced from a))

Exercise 9. *During the lecture, we learned the Jacobian check: For a polynomial map $\varphi : \mathbb{K}^n \to \mathbb{K}^m$, the rank of the Jacobi matrix $J_\varphi(p)$ equals $\dim \operatorname{im}(\varphi)$ for almost all $p \in \mathbb{K}^n$. Now, prove the following:*

a) $\forall p \in \mathbb{K}^n : \operatorname{rank} J_\varphi(p) \leq \dim \operatorname{im}(\varphi)$.

b) *When $\mathbb{K} = \mathbb{R}$ (i.e., φ has real coefficients), then $\dim_{\mathbb{R}} \varphi(\mathbb{R}^n) = \dim_{\mathbb{C}} \varphi(\mathbb{C}^n)$.*

(Hint: Both parts can be deduced from the Jacobian check by combining Exercises 8b and 7.)