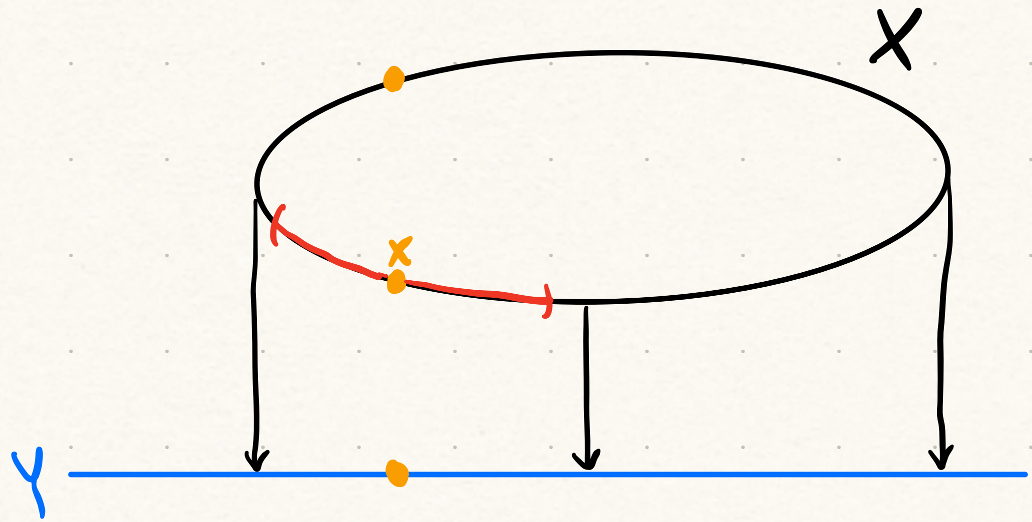
Now let us consider the case of generically finite fibers:

$\varphi: X \to Y$ polynomial map between irreducible varieties, $Y = \overline{\mathrm{im}(\varphi)}$, $\dim(Y) = \dim(X)$.

$\xrightarrow[\text{check}]{\text{Jacobian}}$ For generic $x \in X$, the derivative $d_x\varphi : T_xX \to T_{\varphi(x)}Y$ is bijective.

$\xrightarrow[\text{function thm}]{\text{inverse}}$ For generic $x \in X$, φ is invertible in an open Euclidean neighborhood around x.

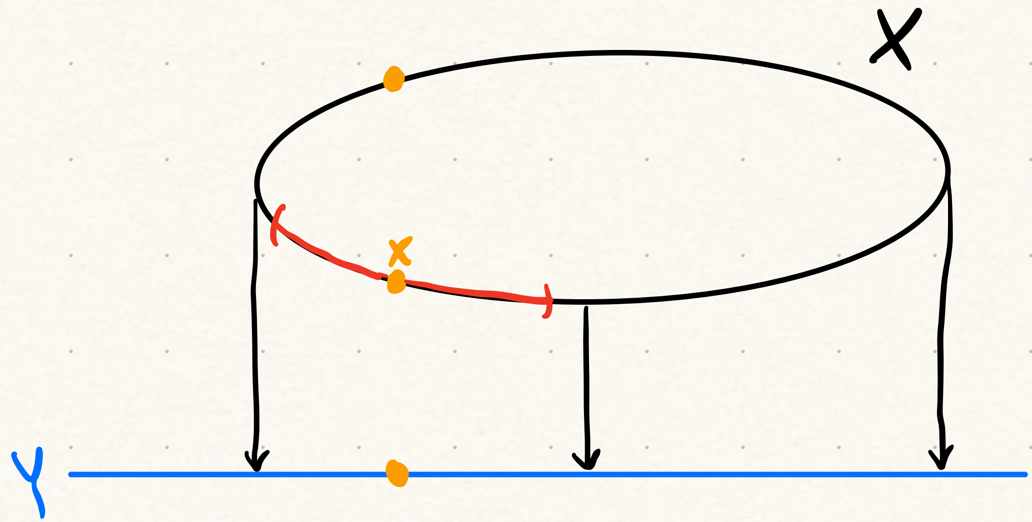


Now let us consider the case of generically finite fibers:

$\varphi: X \to Y$ polynomial map between irreducible varieties, $Y = \overline{\operatorname{im}(\varphi)}$, $\dim(Y) = \dim(X)$.

$\xRightarrow[\text{check}]{\text{Jacobian}}$ For generic $x \in X$, the derivative $d_x\varphi : T_xX \to T_{\varphi(x)}Y$ is bijective.

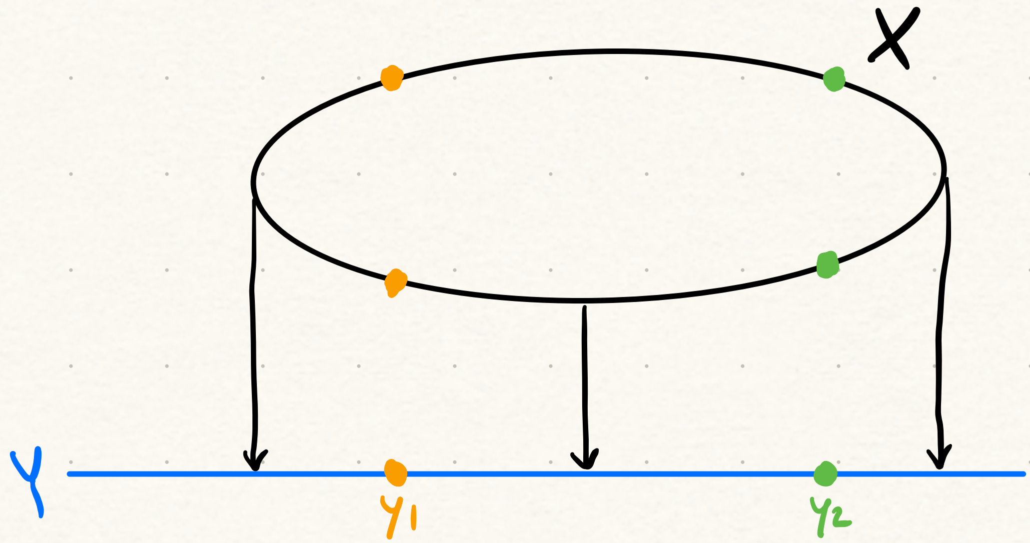
$\xRightarrow[\text{function thm}]{\text{inverse}}$ For generic $x \in X$, φ is invertible in an open Euclidean neighborhood around x.



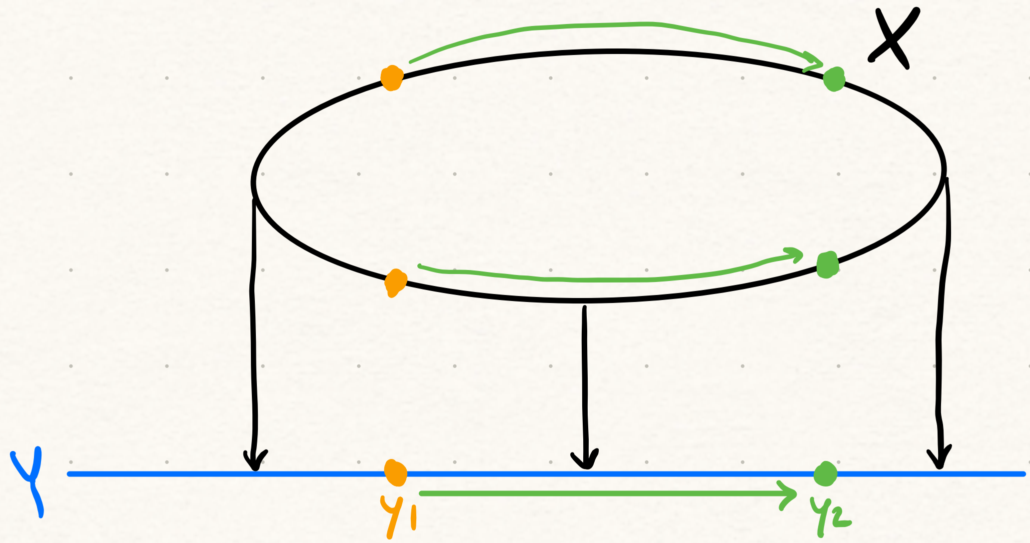
We can exploit this to compute all points in a fiber!

nonlinear inverse problem

E.g., given a function coming from a neural network, find all network parameters that yield the function.

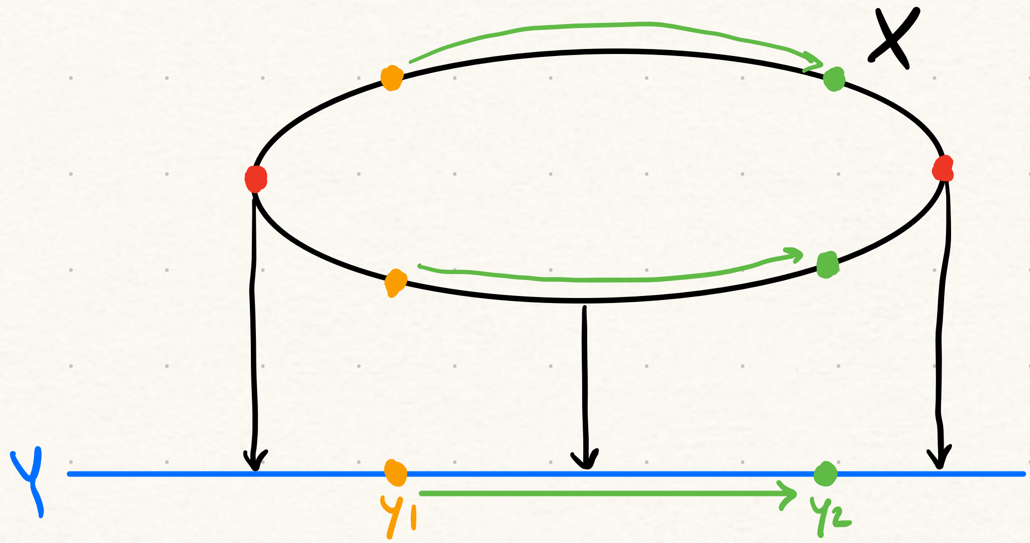


Imagine we know $\varphi^{-1}(y_1)$ and want to compute $\varphi^{-1}(y_2)$.



Imagine we know $\varphi^{-1}(y_1)$ and want to compute $\varphi^{-1}(y_2)$.

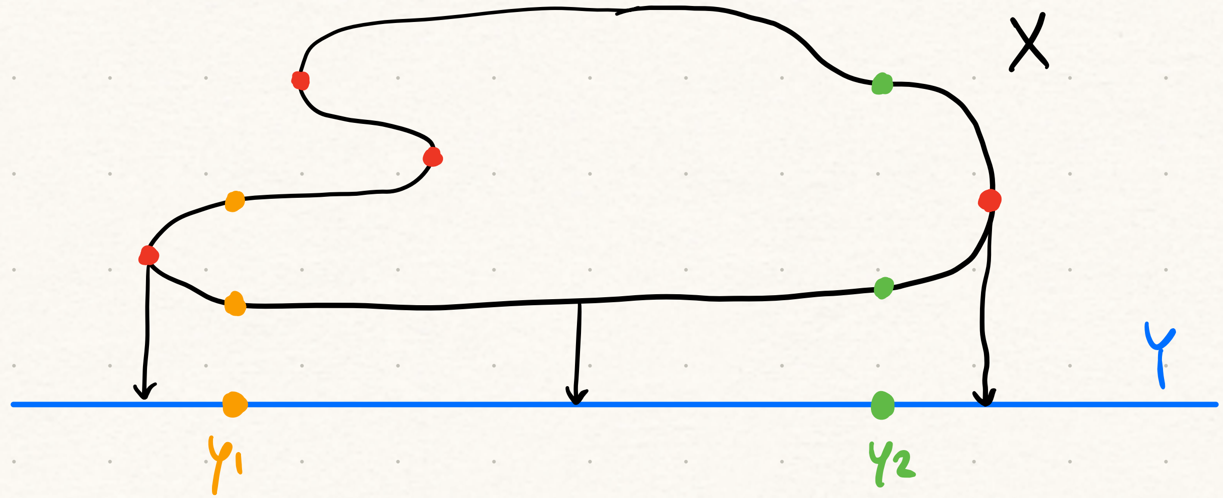
We can track each solution in $\varphi^{-1}(y_1)$ to the solutions in $\varphi^{-1}(y_2)$.



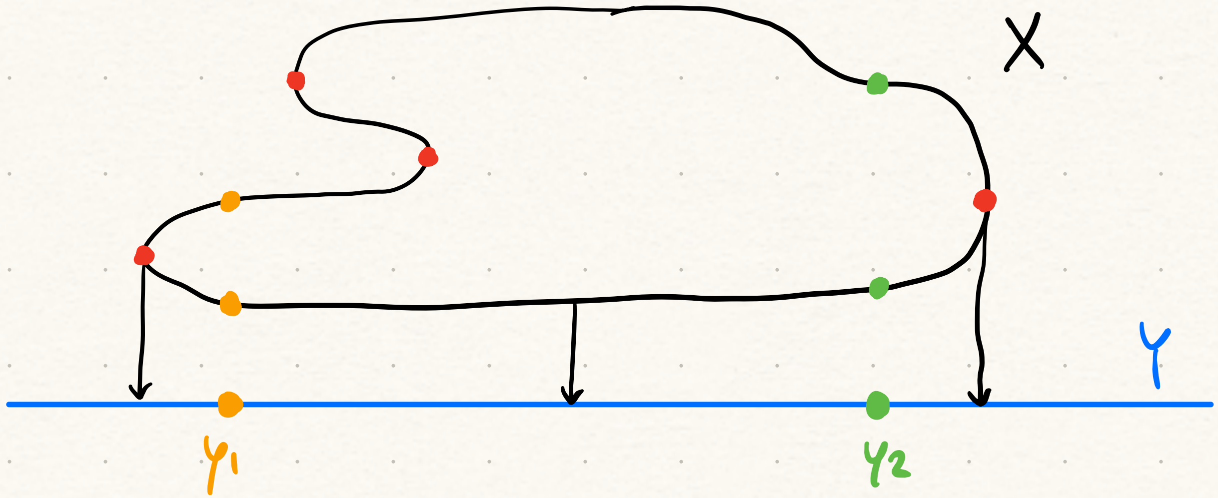
Imagine we know $\varphi^{-1}(y_1)$ and want to compute $\varphi^{-1}(y_2)$.

We can track each solution in $\varphi^{-1}(y_1)$ to the solutions in $\varphi^{-1}(y_2)$.

Problem: φ is <u>not</u> locally invertible at points in the critical locus $\Delta := \{x \in X \mid \operatorname{rank}(d_x(\varphi)) < \dim X = \dim Y\}$.



How can we track from $\varphi^{-1}(y_1)$ to $\varphi^{-1}(y_2)$?



How can we track from $\varphi^{-1}(y_1)$ to $\varphi^{-1}(y_2)$?

over the complex numbers !

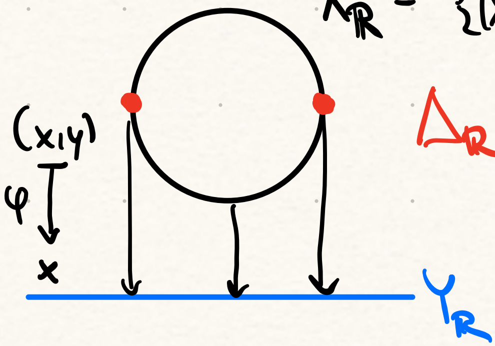
$X \setminus \Delta$ is not connected over $\mathbb{R}$,
but it is connected over $\mathbb{C}$

Reason: $\dim_{\mathbb{C}} \Delta_{\mathbb{C}} < \dim_{\mathbb{C}} X_{\mathbb{C}}$ $\Rightarrow \dim_{\mathbb{R}} \Delta_{\mathbb{C}} \leq \dim_{\mathbb{R}} X_{\mathbb{C}} - 2$

$\Rightarrow X_{\mathbb{C}} \setminus \Delta_{\mathbb{C}}$ is connected

Example:

$$X_{\mathbb{R}} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$$



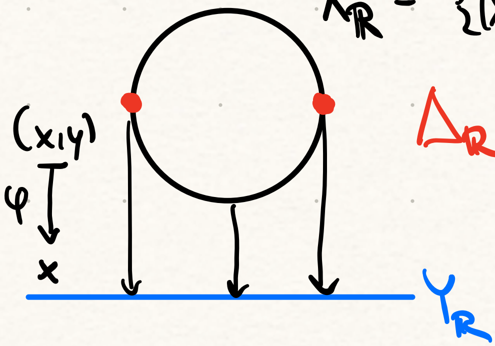
$$\Delta_{\mathbb{R}} = \{(1,0),\ (-1,0)\}$$

$$X_{\mathbb{C}} = \{(x,y) \in \mathbb{C}^2 \mid x^2+y^2=1\}$$

$$\Delta_{\mathbb{C}} = \ ?$$

Example:

$$X_{\mathbb{R}} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$$



$$\Delta_{\mathbb{R}} = \{(1,0),\ (-1,0)\}$$

$$X_{\mathbb{C}} = \{(x,y) \in \mathbb{C}^2 \mid x^2+y^2=1\} \qquad \Delta_{\mathbb{C}} = \{(1,0),\ (-1,0)\}$$

$$T_{(x,y)} X_{\mathbb{K}} = \{(a,b) \in \mathbb{K}^2 \mid xa+yb=0\} \xrightarrow{d_{(x,y)}\varphi} \mathbb{K}$$

$$(a,b) \longmapsto a$$

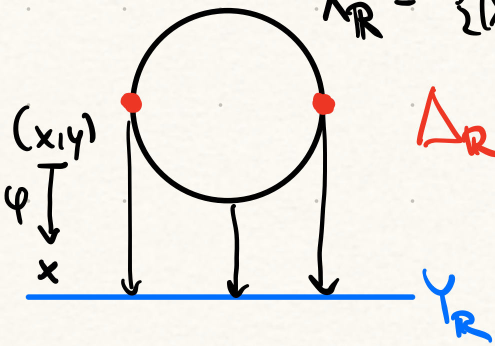
Exercise: $d_{(x,y)}\varphi$ is bijective iff $y \neq 0$

	$\Delta_{\mathbb{C}}$	$X_{\mathbb{C}}$
$\dim_{\mathbb{C}}$		
$\dim_{\mathbb{R}}$		

$$\dim_{\mathbb{R}} \mathbb{C} = 2$$

Example:

$X_{\mathbb{R}} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$



$\Delta_{\mathbb{R}} = \{(1,0),\ (-1,0)\}$

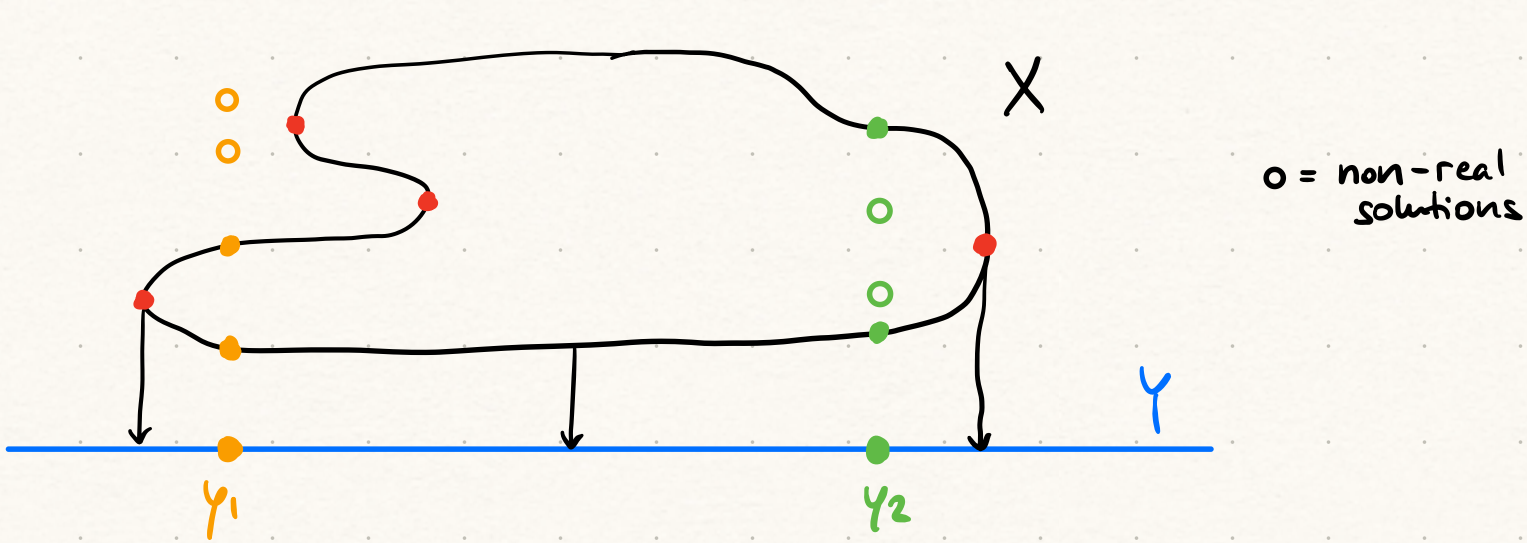
$X_{\mathbb{C}} = \{(x,y) \in \mathbb{C}^2 \mid x^2+y^2=1\}$ $\qquad$ $\Delta_{\mathbb{C}} = \{(1,0),\ (-1,0)\}$

$$T_{(x,y)} X_{\mathbb{K}} = \{(a,b) \in \mathbb{K}^2 \mid xa+yb=0\} \xrightarrow{d_{(x,y)}\varphi} \mathbb{K}$$
$$(a,b) \longmapsto a$$

Exercise: $d_{(x,y)}\varphi$ is bijective iff $y \neq 0$

	$\Delta_{\mathbb{C}}$	$X_{\mathbb{C}}$
$\dim_{\mathbb{C}}$	0	1
$\dim_{\mathbb{R}}$	0	2

$\dim_{\mathbb{R}} \mathbb{C} = 2$

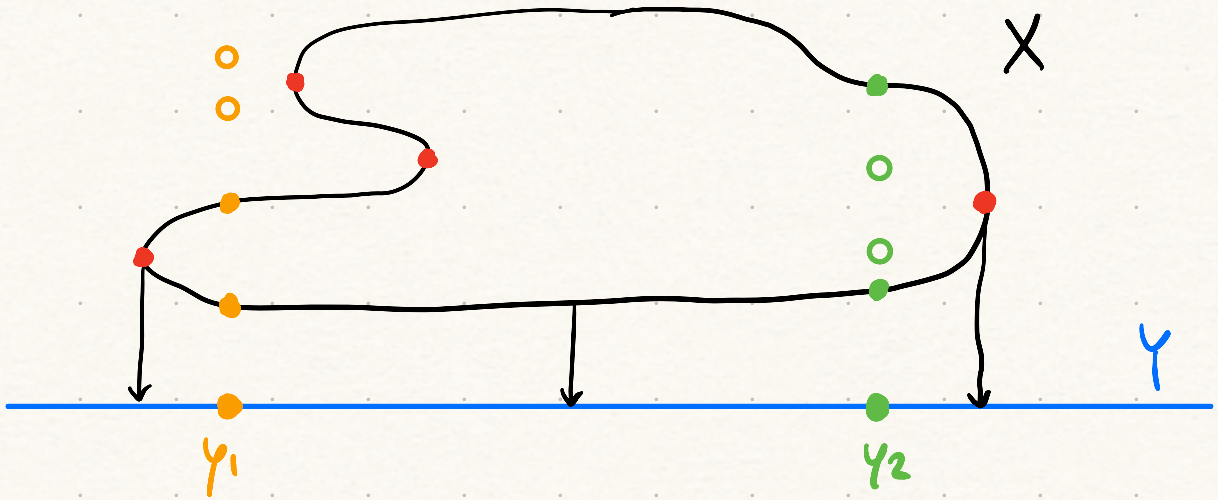


We need to track all complex solutions.

Paul Painlevé (1900): "between 2 truths of the real domain, the easiest & shortest path quite often passes through the complex domain."

This method is called polynomial homotopy continuation.

"Newton's method on speed"



o = non-real solutions

We need to track <u>all</u> complex solutions.

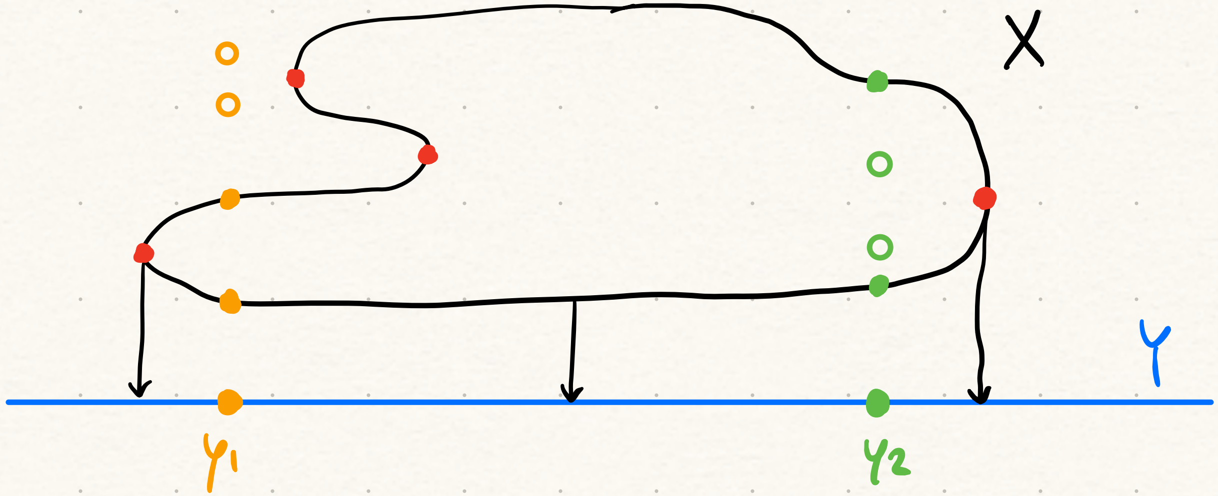
Paul Painlevé (1900): "between 2 truths of the real domain, the easiest & shortest path quite often passes through the complex domain."

This method is called polynomial homotopy continuation.

"Newton's method on speed"

Strategy: ① compute $\varphi^{-1}(y_1)$ for one y_1 somehow

② compute $\varphi^{-1}(y_2)$ quickly for many y_2 using homotopy continuation



We need to track <u>all</u> complex solutions.

> Paul Painlevé (1900): "between 2 truths of the real domain, the easiest & shortest path quite often passes through the complex domain."

This method is called **polynomial homotopy continuation.**

"Newton's method on speed"

Strategy: ① compute $\varphi^{-1}(y_1)$ for one y_1 somehow

② compute $\varphi^{-1}(y_2)$ quickly for many y_2 using homotopy continuation

Exercise: complete the $3 \times 4 \times 2$ tensor

-32	72	-104	
		40	16
-24			0

	-57	-11	-1
10	27	1	
			-7

to a rank-2 tensor using

HomotopyContinuation.jl

How to solve the start system $\varphi^{-1}(y_1)$?

① Gröbner basis: classical symbolic method

can be very computationally expensive

② Monodromy: might not find all solutions

③ Take any other start system that has at least as many solutions as the fibers of φ

⇝ homotopy continuation can track between seemingly different polynomial systems

⇝ HomotopyContinuation.jl can pick a start system for you.

often picks a start system with way too many solutions

⇝ need to track many paths!

7

$$f = \begin{bmatrix} x^2 + 2y \\ y^2 - 2 \end{bmatrix}$$

Solving the equation $f = 0$ can be accomplished as follows

```
using HomotopyContinuation # load the package into the current Julia session
@var x y; # declare the variables x and y
f = System([x^2 + 2y, y^2 - 2]) # construct system f
result = solve(f) # solve f
```

After the computation has finished, you should see the following output.

```
Result with 4 solutions
=======================
• 4 paths tracked
• 4 non-singular solutions (2 real)
• random seed: 0x09c7d125
• start_system: :polyhedral
```

We see that f has two real zeros. They are

```
julia> realsolutions(result)
2-element Array{Array{Float64,1},1}:
 [1.68179, -1.41421]
 [-1.68179, -1.41421]
```

Polynomial Homotopy Continuation - More Formal

Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{p} = (p_1, \dots, p_k)$,

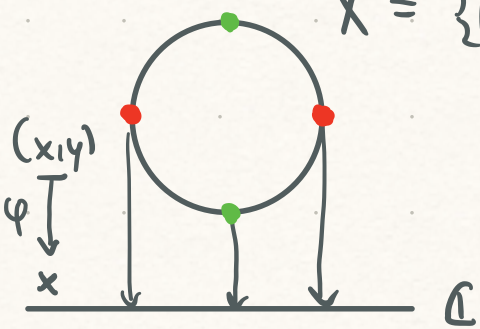
variables parameters

$$\mathcal{F} = \{f_1(\mathbf{x}, \mathbf{p}), \dots, f_n(\mathbf{x}, \mathbf{p})\} \subseteq \mathbb{C}[x_1, \dots, x_n, p_1, \dots, p_k]$$

parametrized family of square polynomial systems.

Example: Computing generically finite fibers of polynomial maps

$$X = \{(x,y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\}$$



x - parameter $(k=1)$

y - variable $(n=1)$

$f_1 = x^2 + y^2 - 1$

Polynomial Homotopy Continuation - More Formal

Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{p} = (p_1, \dots, p_k)$,

variables   parameters

$$\mathcal{F} = \{ f_1(\mathbf{x},\mathbf{p}), \dots, f_n(\mathbf{x},\mathbf{p}) \} \subseteq \mathbb{C}[x_1, \dots, x_n, p_1, \dots, p_k]$$

parametrized family of square polynomial systems.

Definition: Fix $p \in \mathbb{C}^k$.

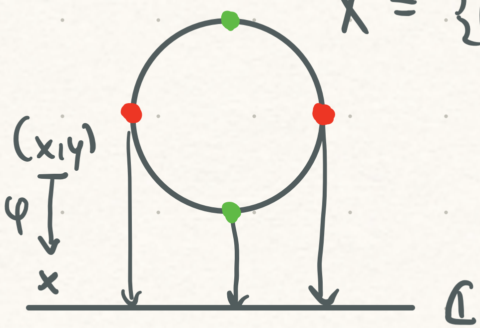
$z \in \mathbb{C}^n$ is a regular zero of $\mathcal{F}$ at p if

1) $f_1(z,p) = \dots = f_n(z,p) = 0$ &

2) $$\det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} (z,p) \neq 0$$

Example: Computing generically finite fibers of polynomial maps

$X = \{(x,y) \in \mathbb{C}^2 \mid x^2+y^2=1\}$



x - parameter $(k=1)$

y - variable $(n=1)$

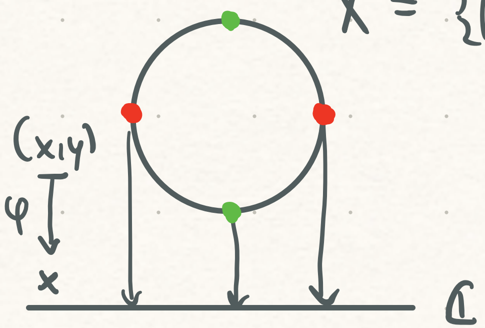
$f_1 = x^2+y^2-1$

y is a regular zero at x if $x^2+y^2=1$ &

$$0 \neq \left[\frac{\partial f_1}{\partial y}\right] = \left[2y\right]$$

Example: Computing generically finite fibers of polynomial maps

$$X = \{(x,y) \in \mathbb{C}^2 \mid x^2+y^2=1\}$$



x - parameter $(k=1)$

y - variable $(n=1)$

$f_1 = x^2+y^2-1$

y is a regular zero at x if $x^2+y^2=1$ &

$$0 \neq \left[\frac{\partial f_1}{\partial y}\right] = [2y]$$

The only non-regular solutions are $y=0$ at $x=\pm 1$.

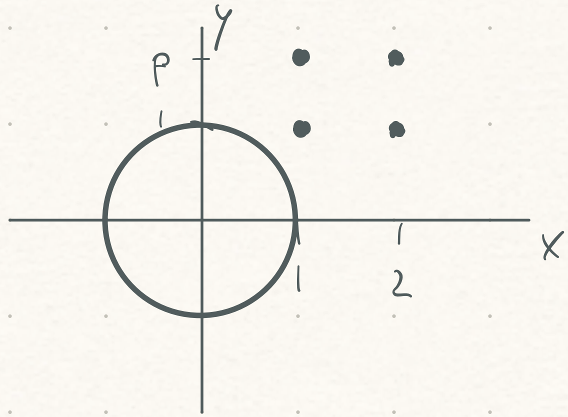
variables $(n=2)$ parameter $(k=1)$

Example: $$f_1(\underbrace{x,y}_{\text{variables } (n=2)};p) = (x-1)\cdot(x-2)\cdot(x^2+y^2-1)$$
$$f_2(x,y;p) = (y-1)\cdot(y-p)\cdot(x^2+y^2-1)$$

has solutions

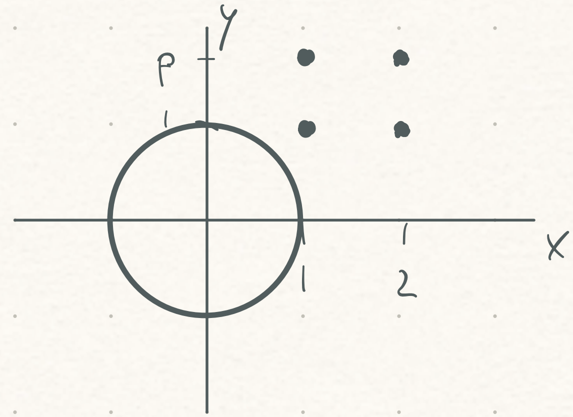
Example: $$f_1(\overbrace{x, y}^{\text{variables } (n=2)}; \overbrace{p}^{\text{parameter } (k=1)}) = (x-1)\cdot(x-2)\cdot(x^2+y^2-1)$$
$$f_2(x, y; p) = (y-1)\cdot(y-p)\cdot(x^2+y^2-1)$$

has solutions



Example: $$f_1(\overbrace{x, y}^{\text{variables } (n=2)}; \overset{\text{parameter } (k=1)}{p}) = (x-1)\cdot(x-2)\cdot(x^2+y^2-1)$$
$$f_2(x, y; p) = (y-1)\cdot(y-p)\cdot(x^2+y^2-1)$$

has solutions



Exercise: a) solutions on the circle $(x^2+y^2=1)$ are never regular

$\Rightarrow$ $(1,p)$ is not regular for $p=0$ &

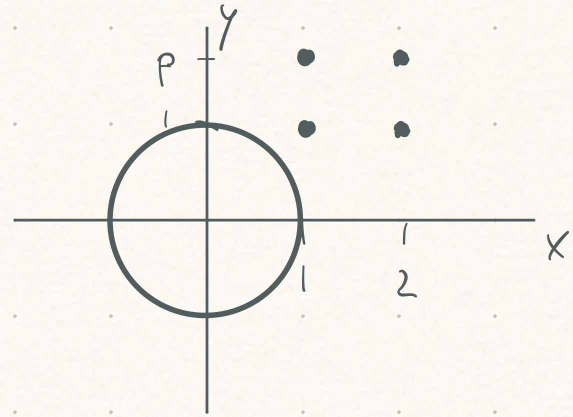
$(2,p)$ is not regular for $p=\pm\sqrt{-3}$

Example:

$$f_1(\overbrace{x,y}^{\text{variables } (n=2)};\overset{\text{parameter } (k=1)}{p}) = (x-1)\cdot(x-2)\cdot(x^2+y^2-1)$$

$$f_2(x,y;p) = (y-1)\cdot(y-p)\cdot(x^2+y^2-1)$$

has solutions



Exercise: a) solutions on the circle $(x^2+y^2=1)$ are never regular

$\Rightarrow$ $(1,p)$ is not regular for $p=0$ &

$(2,p)$ is not regular for $p=\pm\sqrt{-3}$

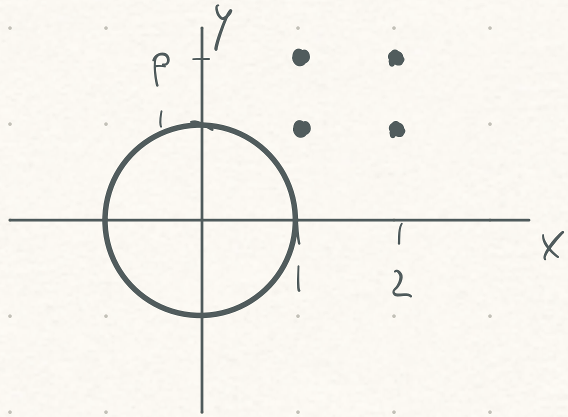
b) no regular zeros for $p=1$

Example:

$$f_1(\overbrace{x,y}^{\text{variables } (n=2)};\overset{\text{parameter } (k=1)}{p}) = (x-1)\cdot(x-2)\cdot(x^2+y^2-1)$$

$$f_2(x,y;p) = (y-1)\cdot(y-p)\cdot(x^2+y^2-1)$$

has solutions



Exercise: a) solutions on the circle $(x^2+y^2=1)$ are never regular

$\Rightarrow$ $(1,p)$ is not regular for $p=0$ &

$(2,p)$ is not regular for $p=\pm\sqrt{-3}$

b) no regular zeros for $p=1$

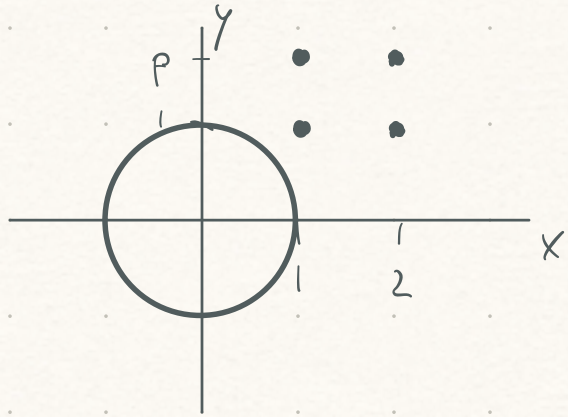
c) 3 regular zeros for $p=0$ or $p=\pm\sqrt{-3}$

variables $(n=2)$ parameter $(k=1)$

Example: $f_1(\underbrace{x,y}; p) = (x-1)\cdot(x-2)\cdot(x^2+y^2-1)$

$f_2(x,y;p) = (y-1)\cdot(y-p)\cdot(x^2+y^2-1)$

has solutions



Exercise: a) solutions on the circle $(x^2+y^2=1)$ are never regular

$\Rightarrow (1,p)$ is not regular for $p=0$ &

$(2,p)$ is not regular for $p=\pm\sqrt{-3}$

b) no regular zeros for $p=1$

c) 3 regular zeros for $p=0$ or $p=\pm\sqrt{-3}$

d) 4 regular zeros for $p\notin\{0,1,\pm\sqrt{-3}\}$

Polynomial Homotopy Continuation - More Formal

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{p} = (p_1, \ldots, p_k)$,

variables ← · · · → parameters

$$\mathcal{F} = \{ f_1(\mathbf{x}, \mathbf{p}), \ldots, f_n(\mathbf{x}, \mathbf{p}) \} \subseteq \mathbb{C}[x_1, \ldots, x_n, p_1, \ldots, p_k]$$

parametrized family of square polynomial systems.

For $p \in \mathbb{C}^k$, let $N(p)$ be the number of regular zeros of $\mathcal{F}$ at p.

$$N := \sup_{p \in \mathbb{C}^k} N(p)$$

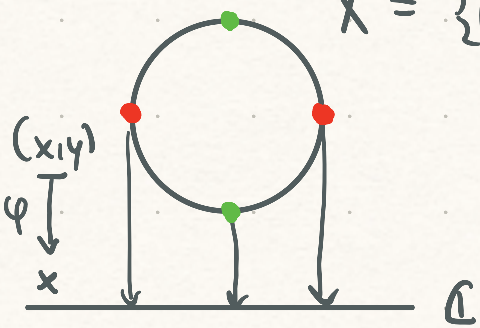
Parameter Continuation Theorem: $N < \infty$ &

there is a proper algebraic subvariety $\Delta \subsetneq \mathbb{C}^k$ such that $N(p) = N$ for all $p \notin \Delta$.

discriminant

Example: Computing generically finite fibers of polynomial maps

$X = \{(x,y) \in \mathbb{C}^2 \mid x^2+y^2=1\}$



x - parameter $(k=1)$

y - variable $(n=1)$

$f_1 = x^2+y^2-1$

y is a regular zero at x if $x^2+y^2=1$ &

$$0 \neq \left[\frac{\partial f_1}{\partial y}\right] = [2y]$$

The only non-regular solutions are $y=0$ at $x=\pm 1$.

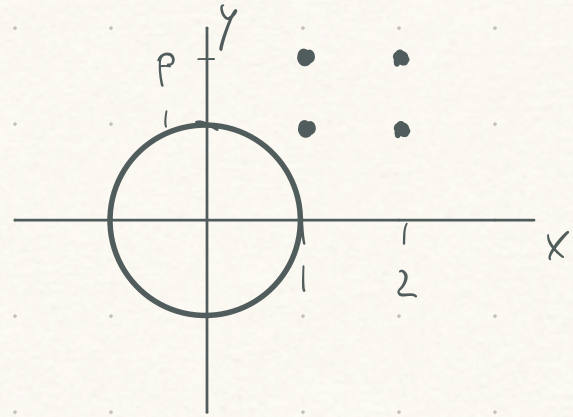
$\Delta = \{-1, +1\}$

$N(p) = 2$ for all $p \notin \Delta$

Example: $f_1(\overbrace{x, y}^{\text{variables } (n=2)}; \overset{\text{parameter } (k=1)}{p}) = (x-1)\cdot(x-2)\cdot(x^2+y^2-1)$

$$f_2(x, y; p) = (y-1)\cdot(y-p)\cdot(x^2+y^2-1)$$

has solutions



Exercise: a) solutions on the circle $(x^2+y^2=1)$ are never regular

$\Rightarrow$ $(1,p)$ is not regular for $p=0$ &

$(2,p)$ is not regular for $p = \pm\sqrt{-3}$

b) no regular zeros for $p=1$

c) 3 regular zeros for $p=0$ or $p=\pm\sqrt{-3}$

d) $\underset{\overset{\shortparallel}{N(p)}}{4}$ regular zeros for $p \notin \{0, 1, \pm\sqrt{-3}\} = \Delta$

Polynomial Homotopy Continuation - More Formal

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{p} = (p_1, \ldots, p_k)$,

variables

parameters

$$\mathcal{F} = \{ f_1(\mathbf{x}, \mathbf{p}), \ldots, f_n(\mathbf{x}, \mathbf{p}) \} \subseteq \mathbb{C}[x_1, \ldots, x_n, p_1, \ldots, p_k]$$

parametrized family of square polynomial systems.

$q \in \mathbb{C}^k \setminus \Delta$ fixed.

$\mathcal{F}(\mathbf{x}, q) = 0$

start system with known zeros

•

•

•

•

$p \in \mathbb{C}^k$ fixed.

$\mathcal{F}(\mathbf{x}, p) = 0$

target system we wish to solve

Polynomial Homotopy Continuation - More Formal

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{p} = (p_1, \ldots, p_k)$,

variables   parameters

$$\mathcal{F} = \{f_1(\mathbf{x},\mathbf{p}), \ldots, f_n(\mathbf{x},\mathbf{p})\} \subseteq \mathbb{C}[x_1,\ldots,x_n, p_1,\ldots,p_k]$$

parametrized family of square polynomial systems.

$$\overbrace{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad}^{\text{piecewise smooth path } \gamma(t) \text{ in } \mathbb{C}^k \text{ with } \gamma(1)=q \ \& \ \gamma(0)=p}$$

$q \in \mathbb{C}^k \setminus \Delta$ fixed.

$\mathcal{F}(\mathbf{x},q) = 0$

start system with known zeros

$p \in \mathbb{C}^k$ fixed.

$\mathcal{F}(\mathbf{x},p) = 0$

target system we wish to solve

⋮

Polynomial Homotopy Continuation – More Formal

Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{p} = (p_1, \dots, p_k)$,

variables parameters

$$\mathcal{F} = \{f_1(\mathbf{x},\mathbf{p}), \dots, f_n(\mathbf{x},\mathbf{p})\} \subseteq \mathbb{C}[x_1, \dots, x_n, p_1, \dots, p_k]$$

parametrized family of square polynomial systems.

$q \in \mathbb{C}^k \setminus \Delta$ fixed.
$\mathcal{F}(\mathbf{x}, q) = 0$
start system with known zeros

piecewise smooth path $\gamma(t)$ in $\mathbb{C}^k$ with $\gamma(1) = q$ & $\gamma(0) = p$

$p \in \mathbb{C}^k$ fixed.
$\mathcal{F}(\mathbf{x}, p) = 0$
target system we wish to solve

track zeros of $\mathcal{F}(x, q) = 0$ to $\mathcal{F}(x, p) = 0$ along the homotopy $H(x,t) := \mathcal{F}(x, \gamma(t))$

Tracking

use numerical algorithms to solve
ordinary differential equation (ODE) initial value problem:

$$\left(\frac{d}{dx} H(x,t)\right) \frac{dx}{dt} + \frac{d}{dt} H(x,t) = 0, \qquad x(1) = z$$

zero of
start system:
$\mathcal{F}(z,q)=0$

Tracking

use numerical algorithms to solve
ordinary differential equation (ODE) initial value problem:

$$\underbrace{\left(\frac{d}{dx} H(x,t)\right)}_{\substack{n\times n \text{ Jacobian matrix:} \\ \text{must stay invertible}}} \frac{dx}{dt} + \frac{d}{dt} H(x,t) = 0, \qquad x(1) = z$$

zero of start system: $\mathcal{F}(z, q) = 0$

Tracking

use numerical algorithms to solve
ordinary differential equation (ODE) initial value problem:

$$\underbrace{\left(\frac{d}{dx} H(x,t)\right)}_{\substack{n\times n \text{ Jacobian matrix:} \\ \text{must stay invertible}}} \frac{dx}{dt} + \frac{d}{dt} H(x,t) = 0 , \qquad x(1) = z$$

zero of start system: $F(z,q)=0$

geometrically:
path $\gamma(t)$ must stay away from discriminant Δ

recall: $\mathbb{C}^k \setminus \Delta$ is connected

Polynomial Homotopy Continuation - More Formal

Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{p} = (p_1, \dots, p_k)$,

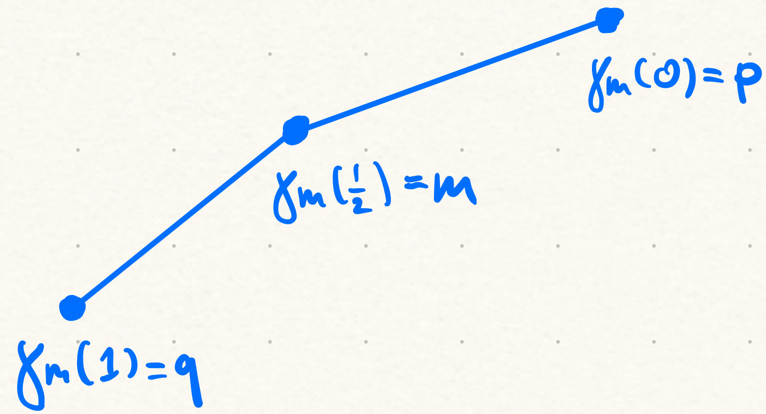
variables   parameters

$$\mathcal{F} = \{ f_1(\mathbf{x}, \mathbf{p}), \dots, f_n(\mathbf{x}, \mathbf{p}) \} \subseteq \mathbb{C}[x_1, \dots, x_n, p_1, \dots, p_k]$$

parametrized family of square polynomial systems.

$$N := \sup_{p \in \mathbb{C}^k} N(p).$$

Fix $q \in \mathbb{C}^k \setminus \Delta$ & $p \in \mathbb{C}^k$.



For $m \in \mathbb{C}^k$, define piecewise linear path

$$\gamma_m(t) := \begin{cases} (2t-1)q + 2(1-t)m, & \text{if } \frac{1}{2} \le t \le 1 \\ 2tm + (1-2t)p, & \text{if } 0 < t \le \frac{1}{2} \end{cases}$$

Polynomial Homotopy Continuation - More Formal

Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{p} = (p_1, \dots, p_k)$,

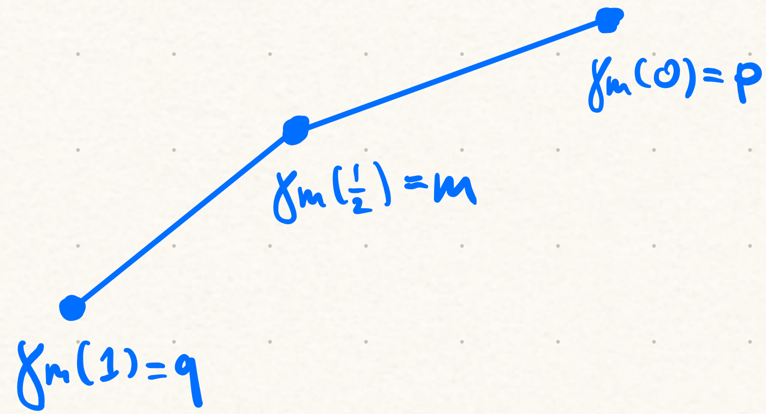
variables, parameters

$$\mathcal{F} = \{f_1(\mathbf{x},\mathbf{p}), \dots, f_n(\mathbf{x},\mathbf{p})\} \subseteq \mathbb{C}[x_1,\dots,x_n, p_1,\dots,p_k]$$

parametrized family of square polynomial systems.

$$N := \sup_{p \in \mathbb{C}^k} N(p).$$

Fix $q \in \mathbb{C}^k \setminus \Delta$ & $p \in \mathbb{C}^k$.



Theorem: For almost all $m \in \mathbb{C}^k$:

1. $\gamma_m((0,1]) \cap \Delta = \emptyset$ "path stays away from discriminant"

Polynomial Homotopy Continuation - More Formal

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{p} = (p_1, \ldots, p_k)$,

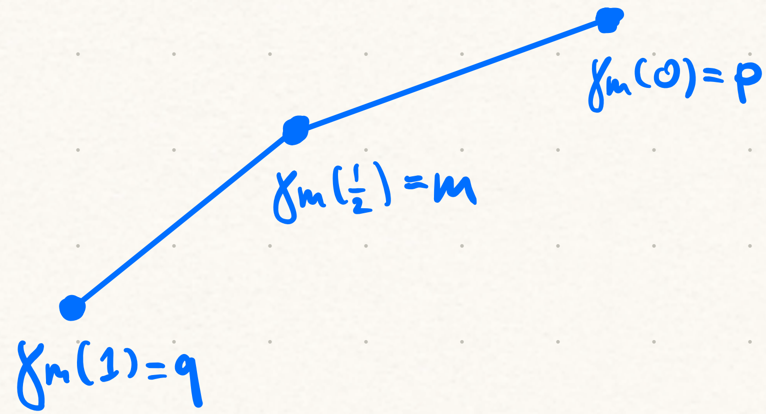
variables parameters

$$\mathcal{F} = \{ f_1(\mathbf{x},\mathbf{p}), \ldots, f_n(\mathbf{x},\mathbf{p}) \} \subseteq \mathbb{C}[x_1, \ldots, x_n, p_1, \ldots, p_k]$$

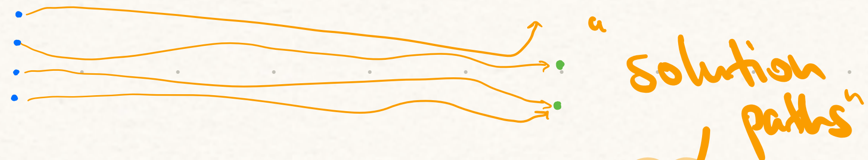
parametrized family of square polynomial systems.

$$N := \sup_{p \in \mathbb{C}^k} N(p).$$

Fix $q \in \mathbb{C}^k \setminus \Delta$ & $p \in \mathbb{C}^k$.



Theorem: For almost all $m \in \mathbb{C}^k$:



② The homotopy $H_m(x,t) := \mathcal{F}(x, \gamma_m(t))$ defines N smooth curves $x(t)$ satisfying $H_m(x(t),t) = 0$ for $0 < t \leq 1$

Polynomial Homotopy Continuation - More Formal

Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{p} = (p_1, \dots, p_k)$,

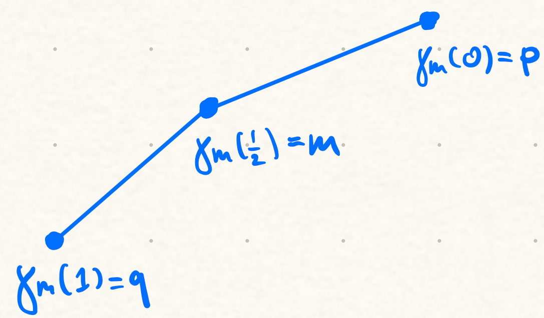
variables

parameters

$$\mathcal{F} = \{f_1(\mathbf{x},\mathbf{p}), \dots, f_n(\mathbf{x},\mathbf{p})\} \subseteq \mathbb{C}[x_1, \dots, x_n, p_1, \dots, p_k]$$

parametrized family of square polynomial systems.

$$N := \sup_{p \in \mathbb{C}^k} N(p).$$



Fix $q \in \mathbb{C}^k \setminus \Delta$ & $p \in \mathbb{C}^k$.

Theorem: For almost all $m \in \mathbb{C}^k$:

3. As $t \to 0$, the limits of the N solution paths include all regular solutions of $\mathcal{F}(x,p) = 0$.
4. If $p \notin \Delta$, every solution path converges to a regular zero of $\mathcal{F}(x,p) = 0$.

How to solve a square system without parameters?

$\mathbf{x} = (x_1, \dots, x_n)$,

$$\boxed{\begin{aligned} f_1(\mathbf{x}) &= 0 \\ &\vdots \\ f_n(\mathbf{x}) &= 0 \end{aligned}}$$

How to solve a square system without parameters?

$\mathbf{x} = (x_1, \dots, x_n)$,
$$\boxed{\begin{aligned} f_1(\mathbf{x}) &= 0 \\ &\vdots \\ f_n(\mathbf{x}) &= 0 \end{aligned}}$$

linearly interpolate between

start system $\mathcal{F}(\mathbf{x}, 1)$ & target system $\mathcal{F}(\mathbf{x}, 0)$

$$\boxed{\begin{aligned} x_1^{\deg(f_1)} - 1 &= 0 \\ &\vdots \\ x_n^{\deg(f_n)} - 1 &= 0 \end{aligned}}$$

How to solve a square system without parameters?

$\mathbf{x} = (x_1, \dots, x_n)$, $\boxed{\begin{array}{l} f_1(\mathbf{x}) = 0 \\ \vdots \\ f_n(\mathbf{x}) = 0 \end{array}}$

linearly interpolate between

start system $\mathcal{F}(\mathbf{x}, 1)$ & target system $\mathcal{F}(\mathbf{x}, 0)$

$$\boxed{\begin{array}{l} x_1^{\deg(f_1)} - 1 = 0 \\ \vdots \\ x_n^{\deg(f_n)} - 1 = 0 \end{array}}$$

↖ will see next lecture: this start system has at least as many zeros as target system & is outside of the discriminant of the interpolating family

Caution

Numerical computations are not exact & can produce errors!

However, there are ways to certify the solutions of polynomial homotopy continuation.

computer proof

Another Issue

If the start system has many solutions, tracking may become very inefficient.

(e.g., start system with 1000 zeros, target system with 8 zeros)

Another Issue

If the start system has many solutions, tracking may become very inefficient.

(e.g., start system with 1000 zeros, target system with 8 zeros)

<u>solution 1:</u>

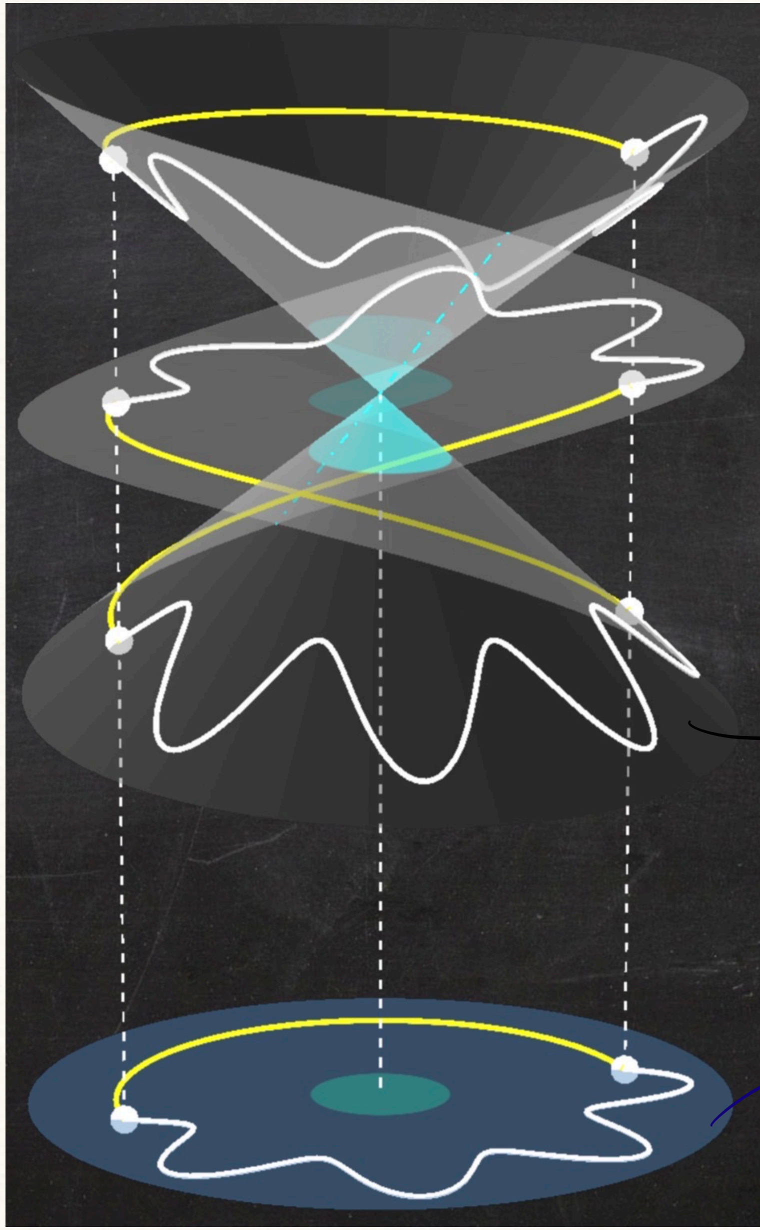
choose a better starting system

(see next lecture)

<u>solution 2:</u>

monodromy

Monodromy



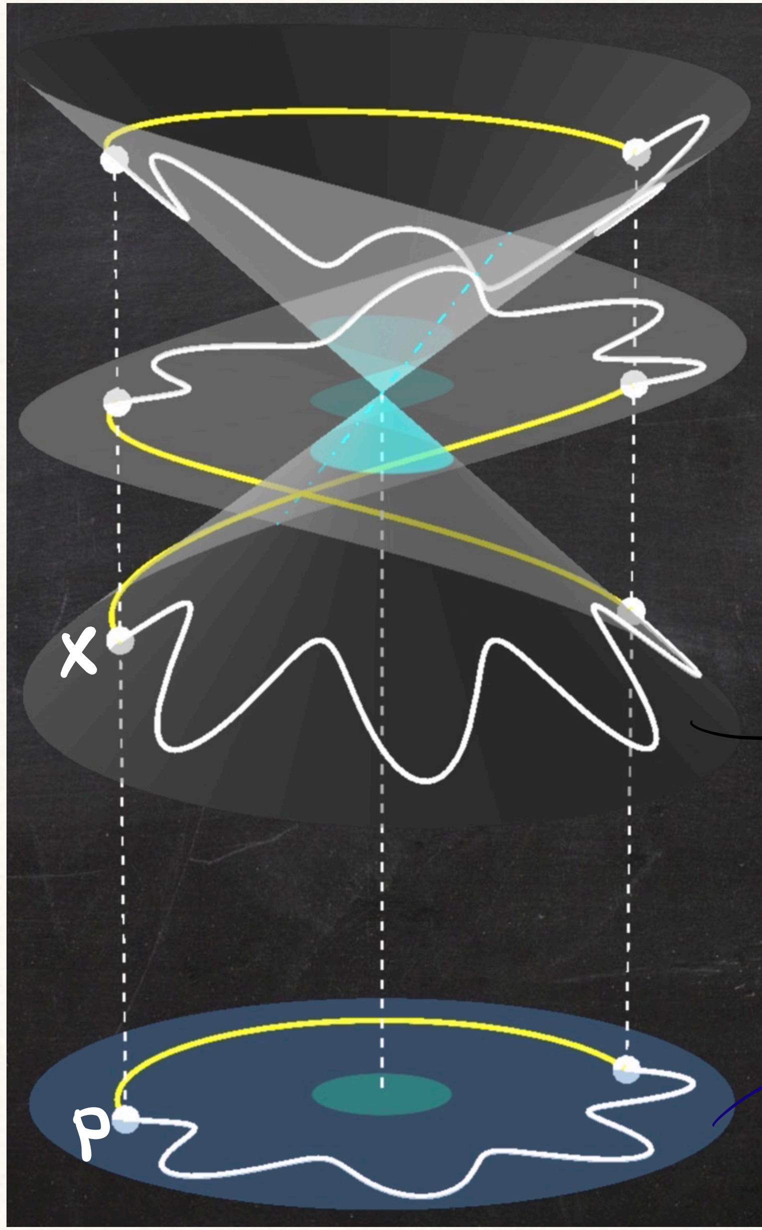
solution variety in $\mathbb{C}^n$

$\mathbb{C}^k$ parameter space

Monodromy

① Find one solution x for some parameter p

(e.g., this is easy if we want to compute fibers of maps $\varphi : X \to Y$: choose a random $x \in X$ and set $p := \varphi(x)$)



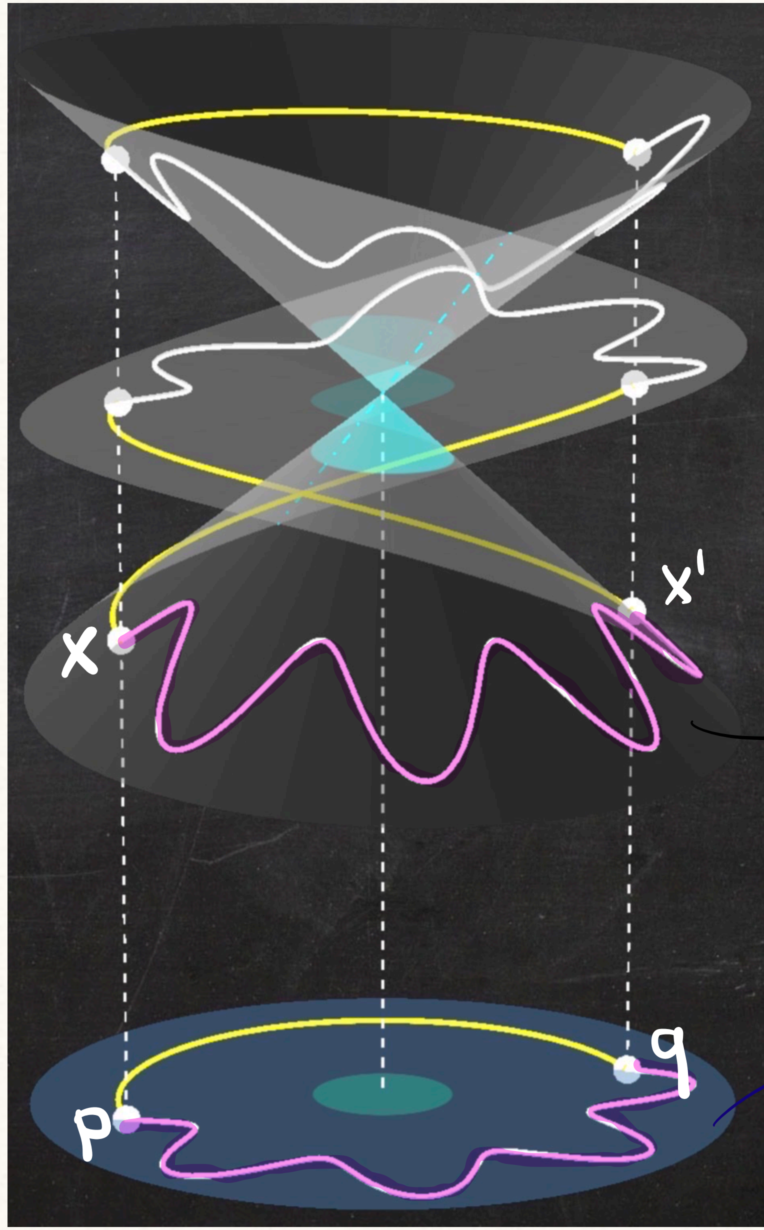
solution variety in $\mathbb{C}^n$

$\mathbb{C}^k$ parameter space

Monodromy

① Find one solution x for some parameter p
(e.g., this is easy if we want to compute fibers of maps $\varphi: X \to Y$: choose a random $x \in X$ and set $p := \varphi(x)$)

② Pick a path from p to another parameter q & track x



solution variety in $\mathbb{C}^n$

$\mathbb{C}^k$ parameter space

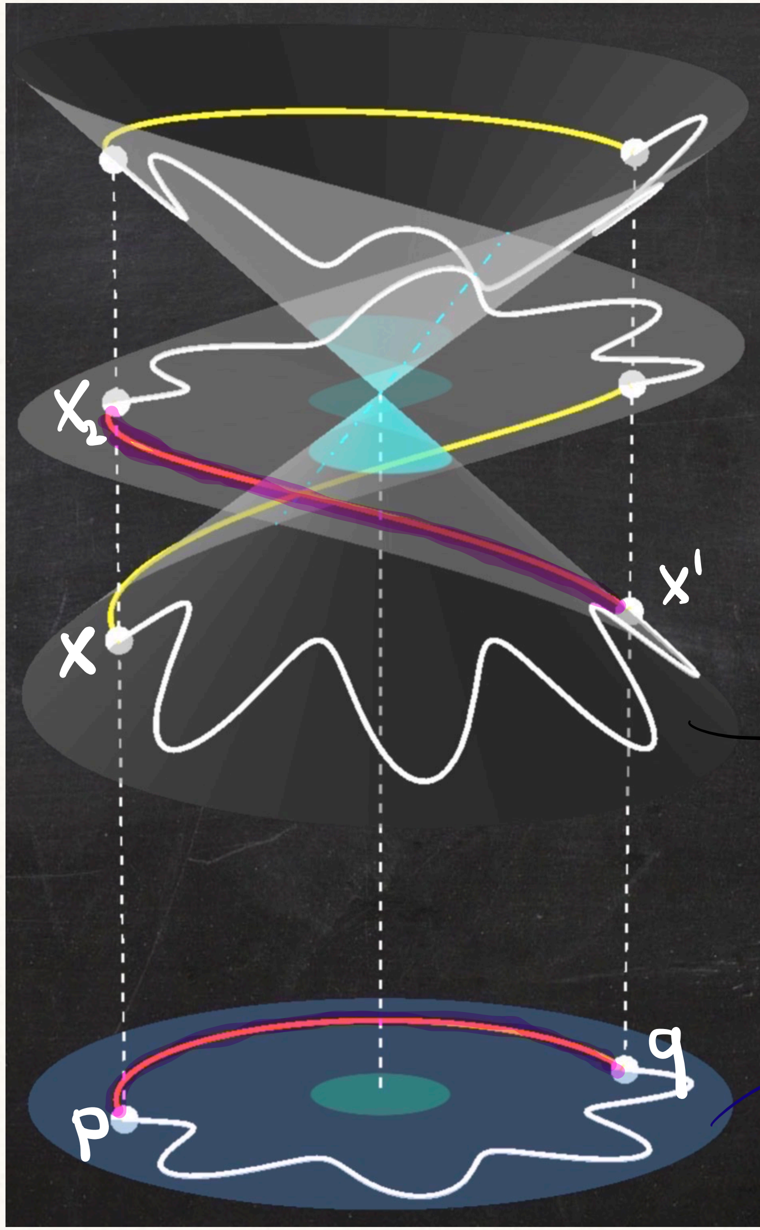
Monodromy

① Find one solution x for some parameter p

(e.g., this is easy if we want to compute fibers of maps $\varphi: X \to Y$: choose a random $x \in X$ and set $p := \varphi(x)$)

② Pick a path from p to another parameter q & track x

③ Pick another path from q to p & track solution



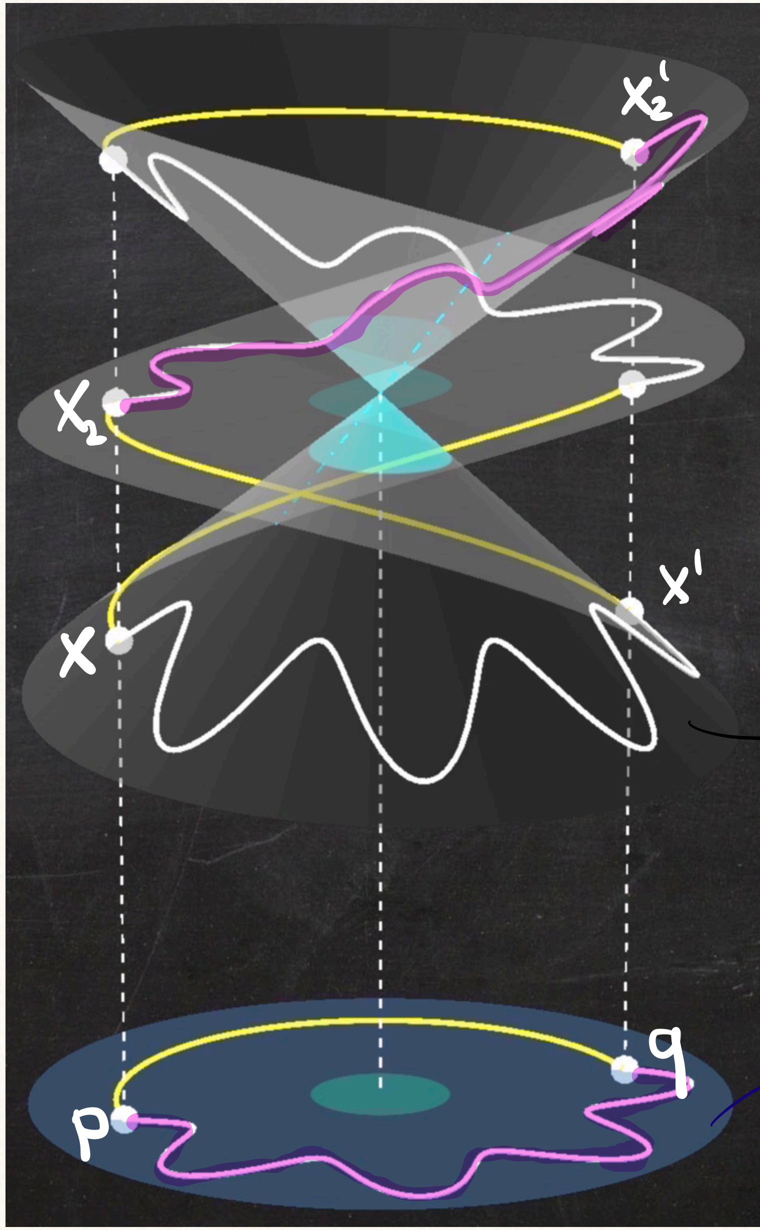
solution variety in $\mathbb{C}^n$

$\mathbb{C}^k$ parameter space

Monodromy

① Find one solution x for some parameter p

(e.g., this is easy if we want to compute fibers of maps $\varphi: X \to Y$: choose a random $x \in X$ and set $p := \varphi(x)$)



solution variety in $\mathbb{C}^n$

$\mathbb{C}^k$ parameter space

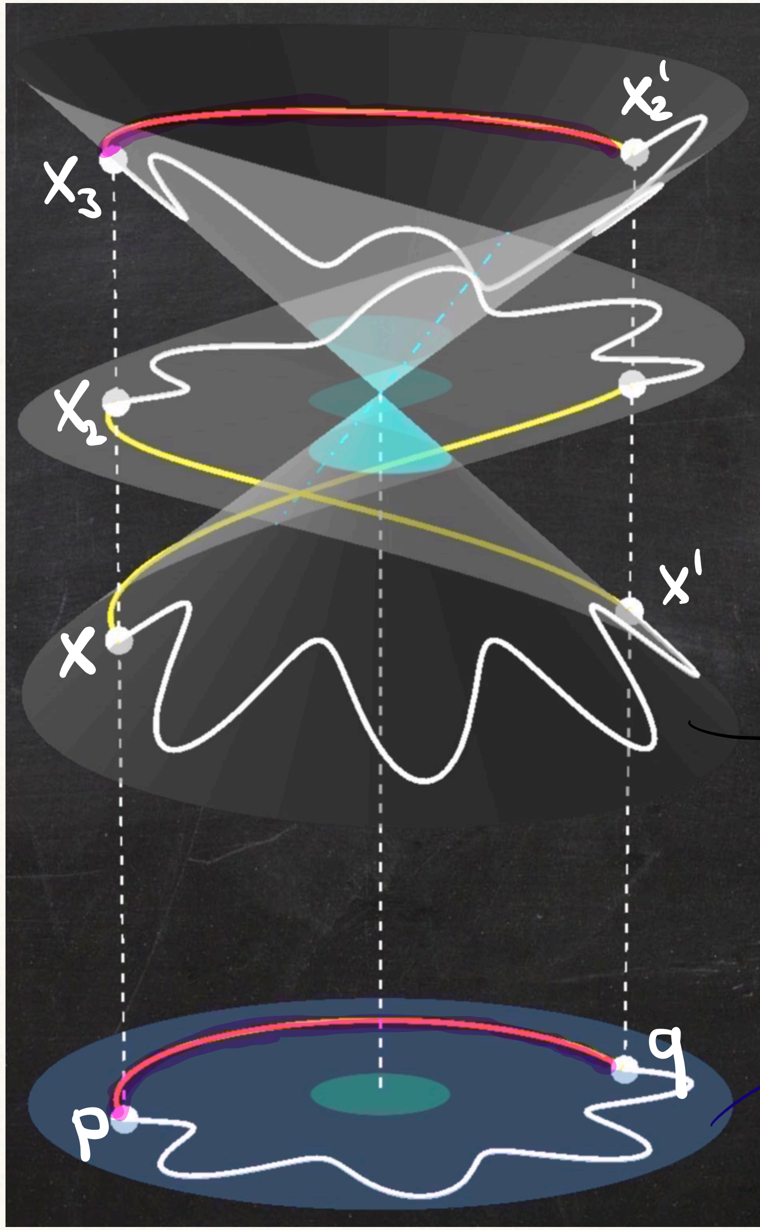
② Pick a path from p to another parameter q & track x

③ Pick another path from q to p & track solution

④ Keep following circular path in $\mathbb{C}^k$ & track solutions until back at x

Monodromy

① Find one solution x for some parameter p
(e.g., this is easy if we want to compute fibers of maps $\varphi: X \to Y$: choose a random $x \in X$ and set $p := \varphi(x)$)



solution variety in $\mathbb{C}^n$

$\mathbb{C}^k$ parameter space

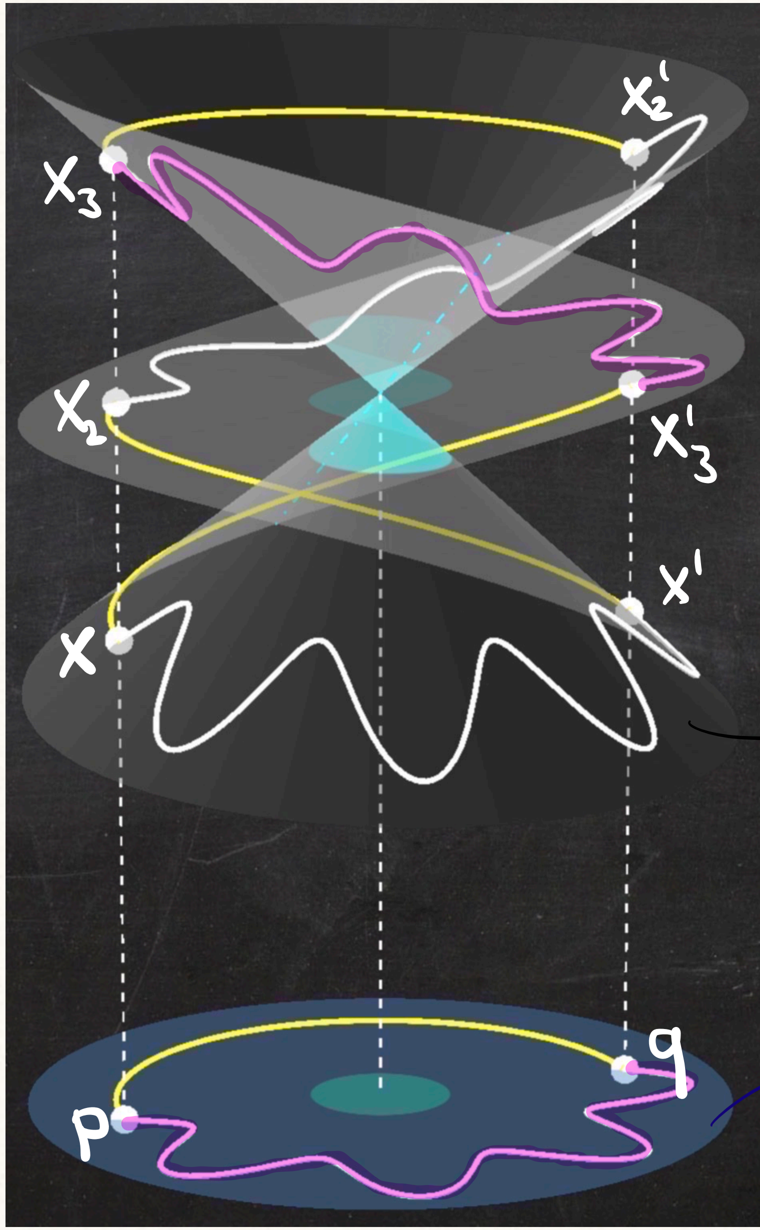
② Pick a path from p to another parameter q & track x

③ Pick another path from q to p & track solution

④ Keep following circular path in $\mathbb{C}^k$ & track solutions until back at x

Monodromy

① Find one solution x for some parameter p
(e.g., this is easy if we want to compute fibers of maps $\varphi: X \to Y$: choose a random $x \in X$ and set $p := \varphi(x)$)



solution variety in $\mathbb{C}^n$

$\mathbb{C}^k$ parameter space

② Pick a path from p to another parameter q & track x

③ Pick another path from q to p & track solution

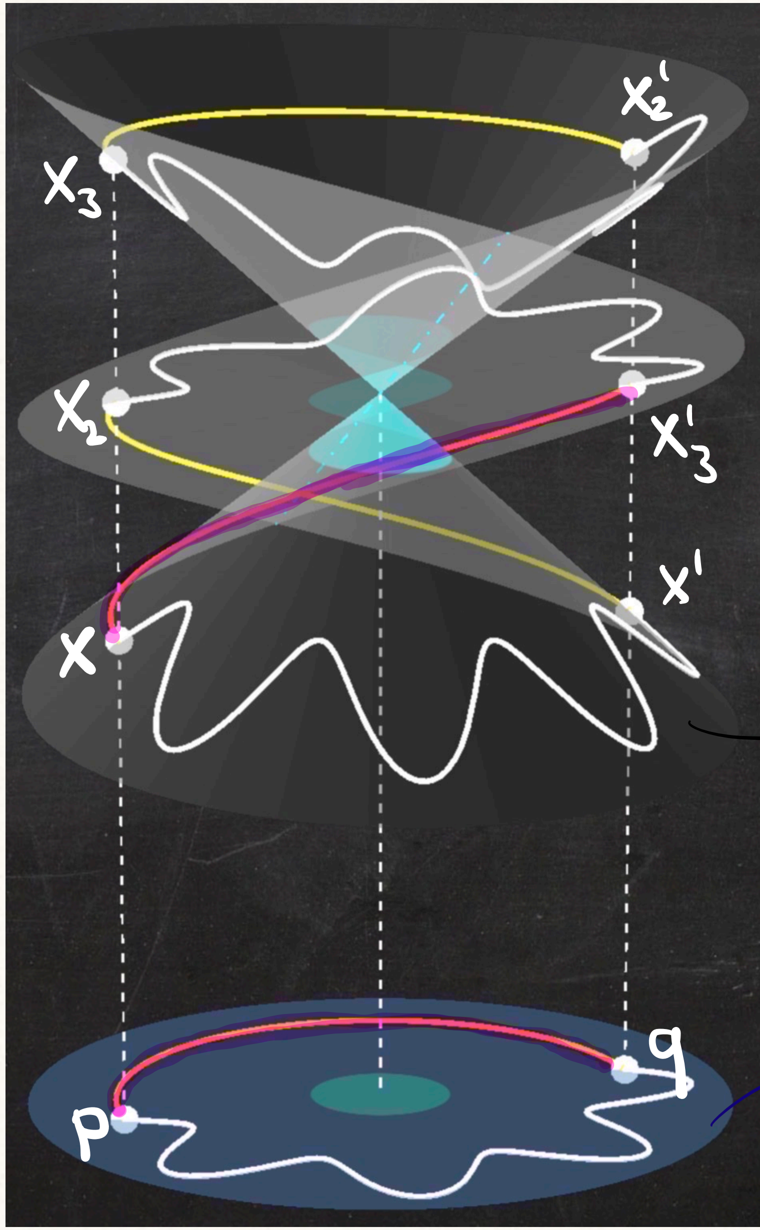
④ Keep following circular path in $\mathbb{C}^k$ & track solutions until back at x

Monodromy

1. Find one solution x for some parameter p
(e.g., this is easy if we want to compute fibers of maps $\varphi: X \to Y$: choose a random $x \in X$ and set $p := \varphi(x)$)

2. Pick a path from p to another parameter q & track x

3. Pick another path from q to p & track solution



solution variety in $\mathbb{C}^n$

$\mathbb{C}^k$ parameter space

4. Keep following circular path in $\mathbb{C}^k$ & track solutions until back at x

Monodromy

applies for instance to maps $\varphi: X \to Y$ between irreducible varities with generically finite fibers

- If the variety $\{(x,p) \in \mathbb{C}^n \times \mathbb{C}^k \mid \tilde{\mathcal{F}}(x,p) = 0\}$ is irreducible, then:

For almost all $p \in \mathbb{C}^k$, monodromy finds - in theory - all solutions x (such that $\mathcal{F}(x,p) = 0$).

☺

Monodromy

applies for instance to maps $\varphi: X \to Y$ between irreducible varities with generically finite fibers

- If the variety $\{(x,p) \in \mathbb{C}^n \times \mathbb{C}^k \mid \tilde{\mathcal{F}}(x,p) = 0\}$ is irreducible, then:

  For almost all $p \in \mathbb{C}^k$, monodromy finds - in theory - all solutions x (such that $\mathcal{F}(x,p) = 0$).

often more computationally intensive than finding the solutions

- In practice, one can use the trace test to verify that all solutions have been found.

Monodromy

applies for instance to maps $\varphi : X \to Y$ between irreducible varities with generically finite fibers

- If the variety $\{(x,p) \in \mathbb{C}^n \times \mathbb{C}^k \mid \tilde{\mathcal{F}}(x,p) = 0\}$ is irreducible, then:

  For almost all $p \in \mathbb{C}^k$, monodromy finds – in theory – **all** solutions x (such that $\mathcal{F}(x,p) = 0$).

often more computationally intensive than finding the solutions

- In practice, one can use the **trace test** to verify that all solutions have been found.

- One can certify the solutions found by monodromy (also against duplicates), which yields a **lower bound** on $N(p)$.

number of solutions

we will learn some upper bounds in the next lecture