Mathematical methods for Economists

Lecture 2: Introduction to functions, linear and quadratic functions

Sofia Tirabassi

Stockholm University

Lecture Goal and Outcome

Goal: Review the concept of one variable function, with special emphasis given to linear and quadratic functions (this is a deepening of concepts you should have seen in the Matte 2b/c classes)

Learning Outcome: At the end of the lecture you will be able to solve problem like the following:

Problem

A firm produces a commodity and receives \$100 for each unit sold. The cost of producing and selling x units is $20x + 025x^2$ dollars.

(a) Find the production level that maximizes profits.

(b) If a tax of \$10 per unit is imposed, what is the new optimal production level?

(c) Answer the question in (b) if the sales price per unit is p, the total cost of producing and selling x units is $\alpha x + \beta x^2$, and the tax per unit is τ where $\tau \leq p - \alpha$.

Why you should care

Mathematical Modelling = Describe the real world with the help of **functions**. If you do not understand functions you cannot understand models (or how they are made).
This is a mathematical model

$$Y = 2{,}262(K)^{0{,}203}(L)^{0{,}763}(1{,}02)^t$$

where

- Y is the national product
- K is the capital invested in stock
- L is the labor
- t is time

Lecture Plan

- Rudiments of set theory
- Intervals & Absolute values
- Functions (graphs, linear and quadratic functions).

Section 1
Intervals and absolute value

Intervals

Let a and b two real numbers with $a \leq b$. We have the finite intervals with extrema a and b

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
  This is the closed interval with extremes a and b.
- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
  This is the open interval with extremes a and b.
- $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

These last intervals are called half-open.
We have also infinite intervals

- $[a, +\infty) = \{x \in \mathbb{R} \mid a \leq x\}$
- $(a, +\infty) = \{x \in \mathbb{R} \mid a < x\}$
- $(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$
- $(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$
- $(-\infty, +\infty) = \mathbb{R}$

Absolute value

Given x a real number we have that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Roughly speaking the absolute value of x is "x without the sign".

- $|-100| =$
- $|1000| =$
- $|\pi| =$

Economic Reality

In economics, we often care about the **magnitude of a deviation or change**, regardless of whether it is positive or negative (e.g., budget deficits vs. surpluses).

Important example

We can use the absolute value to express rooth of powers:

$$\sqrt{a^2} = |a|$$

Questions?

Active Learning 1: Concept Check

Stockholm University

Identify the Target Deviation (1 Minute)

Look at the table below showing budget deviations (x) for four operational units. Which department generated the largest **absolute variance** $|x|$?

Department	**Budget Variance (x)**
Department A	+$10,000
Department B	-$12,000
Department C	+$5,000
Department D	-$3,000

Action: On my count of three, raise 1, 2, 3, or 4 fingers for A, B, C, or D.

2. Theory: The Two-Branch Algebraic Method

Stockholm University

To solve an absolute value inequality, we strip away the absolute value bars by splitting the problem into two distinct paths.

$$|\text{expression}| \geq c$$

- If expression ≥ 0 then expression $\geq c$
- If expression < 0 then -expression $\geq c$

3. Application: Working with Coefficients

In economic applications, variables are regularly weighted by constants (e.g., tax rates, elasticities, processing scaling bounds).

A central logistics firm estimates that twice their daily shipping container variance (x) minus a fixed baseline terminal processing lag of 5 hours must remain within a maximum tolerance of 9 hours to satisfy supply chain penalty thresholds:

$$|2x - 5| \leq 9$$

Let's clear the bars using both branches...

Active Learning : Think-Pair-Share

Stockholm University

Problem: Portfolio Volatility Limits (5 Minutes)

An automated trading platform tracks a leveraged equity asset (p). A flash freeze is automatically triggered if three times the market deviation price minus an institutional liquidity premium of $12 drifts outside safe bounds by more than $18:

$$|3p - 12| > 18$$

Task Instructions:

1. **Think (1 min):** Set up and solve the Positive and Negative branches individually on your scratch paper.
2. **Pair (2 mins):** Verify your endpoints and inequality signs with your neighbor. Did you remember to flip the sign?
3. **Share (2 mins):** We will map the trigger zones together.

Section 2
Functions

Functions

A function f is an application that assign to any element of a set D (called domain of f) a unique element of a set B. The range of f is the set of all possible value that f can take.

Example

It is very common to view many quantities (population, capital, radioactivity) as functions of time.

- Table of a function.
- Graph of a function.
- Sometimes we have formulas (models) for computing the output.

$$K(t) = K_0 \left(1 + \frac{p}{100}\right)^t$$

Mathematical set up

- A function f maps elements from an input set to an output set: $f : X \to Y$.
- **Domain (X):** The set of all valid inputs (e.g., non-negative production levels).
- **Codomain (Y):** The target set where outputs are allowed to land.
- **Range ($f(X)$):** The set of all *actual* values generated by the function.

Key Distinction

The Range is always a subset of the Codomain (Range $\subseteq$ Codomain).

Natural/Maximal domain

Find the (maximal) domain of the following functions

- $f(x) = (\sqrt{1-x} - 2)^{-1}$
- $g(t) = \sqrt{1 - |t|}$

Active Learning 1: Domain Boundaries

Think-Pair-Share: Economic Constraints (3 Minutes)

Consider a firm's total revenue function: $R(q) = 50q$, where q is the quantity of units sold. Mathematically, the domain of any polynomial function is all real numbers ($\mathbb{R}$).

Your Task:

1. **Think (1 min):** Why is the economic domain of $R(q)$ different from its mathematical domain? Write down the true economic domain.
2. **Pair (1 min):** Compare with your neighbor. Does your domain include decimals or fractions? Why?
3. **Share (1 min):** Let's define why **contextual constraints** restrict domains in business forecasting.

Composition

Composition of Functions: We apply several functions one after the other.

Mathematical Formulation

Given two functions $g : X \to Y$ and $f : Y \to Z$, the composite function $(f \circ g)$ is:

$$(f \circ g)(x) = f(g(x))$$

Real-World Example:

- Let $n = g(t)$ be consumer foot traffic as a function of temperature (t).
- Let $R = f(n)$ be store revenue as a function of foot traffic (n).
- The composite function $R(g(t))$ yields revenue directly based on the weather.

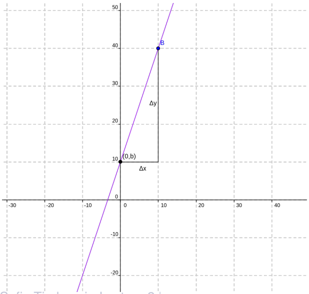
Questions?

Linear functions

$$f(x) = ax + b$$

Where *a* and *b* are two real numbers.

- Domain?
- Range?
- Why linear?



The number *b* is the *y*-intercept.
The number *a* is called the slope. We have that

$$a = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2}$$

Exercise (5 minutes)

The number of employees at a company is predicted to grow linearly the next 10 years. Last year, there were 100 people at the company, and this year, there are 122. How many are predicted to work at the company 6 years from now?

Strategy: find the linear functions that describes the number of employees as a function of time.

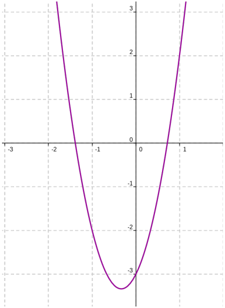
Questions?

Stockholm University

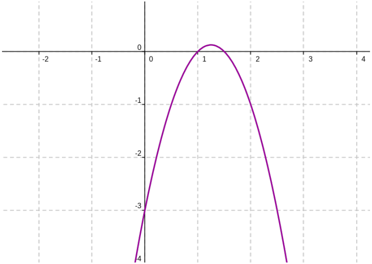
Quadratic functions

$$f(x) = ax^2 + bx + c$$

Where a, b, and c are real numbers, $a \neq 0$.



$a > 0$



$a < 0$

The range of quadratic functions

Completing the square we have that

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a},$$

Thus, we have that $f(x) = ax^2 + bx + c$ has

- a minimum point in $x = -\frac{b}{2a}$, when $a > 0$, and
- a maximum point in $x = -\frac{b}{2a}$, when $a < 0$, and

So we have that

$$R_f = \begin{cases} \left\{y \in \mathbb{R} \mid y \geq -\frac{b^2-4ac}{4a}\right\} & \text{if } a > 0 \\ \left\{y \in \mathbb{R} \mid y \leq -\frac{b^2-4ac}{4a}\right\} & \text{if } a < 0 \end{cases}$$

The zeroes of quadratic functions

Stockholm University

Completing the square we have that

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a},$$

so we have that $f(x) = ax^2 + bx + c = 0$ if, and only if

- $\Delta := b^2 - 4ac \geq 0$, and
-

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Find the zeroes of $f(x) = x^2 - 3x - 2$

Back to our problem

Stockholm University

Problem

A firm produces a commodity and receives \$100 for each unit sold. The cost of producing and selling x units is $20x + 0.25x^2$ dollars.

(a) Find the production level that maximizes profits.

(b) If a tax of \$10 per unit is imposed, what is the new optimal production level?

(c) Answer the question in (b) if the sales price per unit is p, the total cost of producing and selling x units is $\alpha x + \beta x^2$, and the tax per unit is τ where $\tau \leq p - \alpha$.

- Find the profit as a function of the units produced (Hint: it will be a quadratic function).
- Find the maximum of the function.

Questions?

Thank you for your attention!