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Course MM3001

Lecture 3: Other functions

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Lecture Goal and Outcome

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Goal: Review of useful functions often used in Economics (this is a deepening of concepts you should have seen in the Matte 2b/c and Matte 3 classes)

Learning Outcome: After today you will be able to

- Find zeros of polynomial functions
- Solve problems like the following:

Problem

An investment gives 2% interest every year. After how many years has an initial investment doubled? Tripled?

Why you should care

Many man made and naturally occurring phenomena, including city sizes, incomes, word frequencies, and earthquake magnitudes, are distributed according to a power-law distribution.

Example

The probability that a person has an income greater than x is $P(x) = Cx^a$ with C a positive real number and a a negative real number.

You have a set of data and you want to find C and a that fits the data best. Apply the logarithm on the data set and you will see that your data now fit in a line (and you can use linear regression).

$$\log(P(x)) = a \log(x) + \log(C),$$

Why you should care

NOT EVERYTHING IS A LINE

Consider these three economic phenomena:

1. **Uber Ride Costs:** A passenger pays a fixed base fee plus a constant amount for every kilometer traveled.
2. **Diminishing Returns:** Adding more workers initially boosts factory output a lot, but eventually, output levels off due to crowding.
3. **Compound Interest:** A savings account balance grows slowly at first, but spirals upwards faster and faster every year.

Your Task:

- **Think (1 min):** Which of these can be modeled by a straight line? Which ones need a curved function (like an arch or a steep ramp)?
- **Pair (2 mins):** Compare notes with your neighbor. Why is a line a bad tool for modeling viral growth or market saturation?

Lecture Plan

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- Polynomials
- Power functions
- Exponential functions
- Logarithmic functions

Section 1
Polynomials

Polynomial functions

A polynomial function is a function of the form

$$p(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0,$$

where d is a natural number and the coefficient c_i's are real numbers.

- If d is the highest power appearing in the law of p we say that p is a polynomial of degree d (we write $\deg(p) = d$).
- The monomial of highest degree $c_d x^d$ is called the leading term of p.
- The coefficient c_d is called the leading coefficient of p.
- The coefficient c_0 is called the constant coefficient of p.

Convention

The identically 0 function ($0(x) = 0$) is a polynomial and we set $\deg(0) = -\infty$.

Examples

- Linear functions are polynomial functions. What is their degree?
- Quadratic functions are polynomial functions. What is their degree?
- Constant functions are polynomial functions. What is their degree?
- $x^3 - 1$
- $x^{1000} + 2x^{100} + 3x$

Polynomial multiplication

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If one multiplies two polynomial functions p and q, one gets again a polynomial function.

Example

Multiply $(x+3)^2$ and $(x+2)$.

Useful fact:

$$\deg(pq) = \deg(p) + \deg(q),$$

Example

1. $(x^2 - 1) = (x + 1)(x - 1)$.
2. $(x^3 - 1) = (x^2 + x + 1)(x - 1)$.
3. $(x^4 - 1) = (x^3 + x^2 + x + 1)(x - 1)$.
4. $(x^n - 1) = (x^{n-1} + x^{n-2} + \ldots + x + 1)(x - 1)$.

Roots

We say that a polynomial p has a root or zero in the point $a \in \mathbb{R}$ if $p(a) = 0$.

- Polynomials of odd degree always have at least a real root.
- Polynomials of even degree might or not have real roots.

Problem

How do we find the roots?

Finding roots of polynomials and in general zeroes of functions will be really important when we will speak about optimization and study of functions (Lecture 7 and usually an exam problem).

If a polynomial is of degree 2, then

$$p(x) = ax^2 + bx + c$$

with $a \neq 0$ and we can find roots by quadratic formula:

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Higher Degree

For polynomial of degree 3 and 4 there are also (horrible) formulas. Given a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, the roots x_k can be expressed as:

$$x_k = -\frac{1}{4a}B + \frac{1}{2a}\sqrt{D}\cos\left(\frac{\theta + 2k\pi}{4}\right) - \frac{1}{2a}\sqrt{-D}\sin\left(\frac{\theta + 2k\pi}{4}\right)$$

where:

$$B = b - \frac{3c^2}{8a}$$

$$D = c^2 - \frac{4bd}{4a} + \frac{12cd}{16a^2} - \frac{64ace}{256a^3}$$

$$\theta = \arccos\left(\frac{4ac - b^2}{4\sqrt{a^2(c^2 - \frac{4bd}{4a} + \frac{12cd}{16a^2} - \frac{64ace}{256a^3})}}\right)$$

Here, $k = 0, 1, 2, 3$ for the four roots.

Higher Degree

For degree 5 and above...there is not even a formula! What then?

- There are methods to give an approximation of the roots.(They typically use linear algebra, differential calculus, integral calculus).
- There are ways (that do not always work) to find “easy” roots, and use them to factor the polynomial as a product of polynomials of lower degree.

Questions?

Active Learning: Marginal Cost vs. Profit Parabolas

Think-Pair-Share: Identifying Polynomial Economic Roles (3 mins)

A firm operating in the Swedish market analyzes two internal key metrics:

- **Function 1:** $C(q) = 250q + 45\,000$ (Total Cost)
- **Function 2:** $\Pi(q) = -2q^2 + 120q - 800$ (Net Profit)

Instructions:

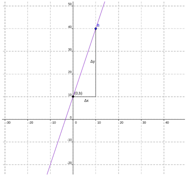
1. **Think (1 min):** What is the mathematical degree of each function? What does the constant 45 000 SEK represent regarding the firm's fixed structures?
2. **Pair (1 min):** Compare leading coefficients. Why is the leading coefficient in the profit function negative (-2)? What does this tell you about finding a maximum?
3. **Share (1 min):** How does this connect to the economic concept of diminishing returns?

Section 2
Power Functions

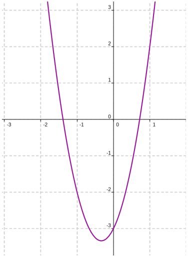
Increasing/Decreasing functions

- A function $f(x)$ is said to be increasing on an interval I if for every x and $y \in I$ we have that $x < y$ implies that $f(x) \leq f(y)$
- A function $f(x)$ is said to be strictly increasing on an interval I if for every x and $y \in I$ we have that $x < y$ implies that $f(x) < f(y)$
- A function $f(x)$ is said to be decreasing on an interval I if for every x and $y \in I$ we have that $x < y$ implies that $f(x) \geq f(y)$
- A function $f(x)$ is said to be strictly decreasing on an interval I if for every x and $y \in I$ we have that $x < y$ implies that $f(x) > f(y)$

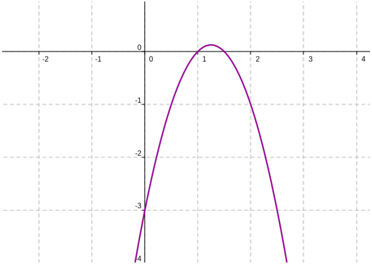
Examples



$3x + 10$



$a > 0$



$a < 0$

Questions?

Power functions

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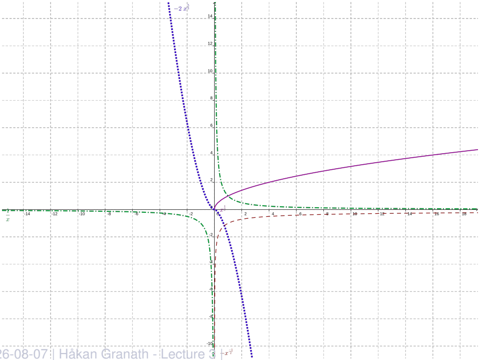
A power function is a function of the form

$$f(x) = Cx^q$$

Where C is a real number and q is a rational number.



- The domain can be either $\mathbb{R}$, $\mathbb{R}^*$, $\mathbb{R}_{\geq 0}$ or $\mathbb{R}^+$, depending on q.

Questions?

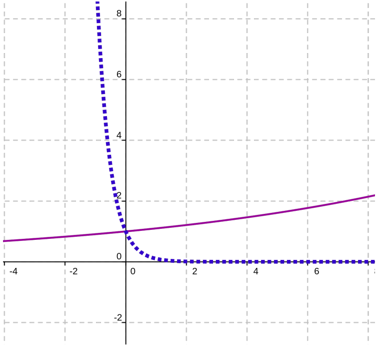
Section 3
Exponential functions

Exponential function

An exponential function is a function of the form

$$f(x) = Ca^x$$

Where C is a real number and a is a positive real number.



- The domain is all $\mathbb{R}$
- Different behavior if $a > 1$ or $a < 1$.
- They are always increasing or decreasing.
- They have the same positivity of C

The number e

Suppose you invest a capital K for 1 year at interest rate x. After one year you have

$$K\left(1+\frac{x}{1}\right)^{1},$$

If you disinvest after 6 month and reinvest immediately you will have

$$K\left(1+\frac{x}{2}\right)^{2},$$

With n withdrawal and deposit:

$$K\left(1+\frac{x}{n}\right)^{n} \to Ke^{x},$$

We will see that this number is really useful when integrating and deriving. Thus e^x is the best exponential function.

From a to e

The range of e^x is the whole positive real line. So if a is a positive real number we can write $a = e^r$ for some r. Thus

$$f(x) = Ca^x = C(e^r)^x = Ce^{rx},$$

Exercise

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Find a and C such that $f(x) = Ca^x$ fits the data points $(-1, 2/3)$ and $(3, 54)$

Questions?

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Section 4
Logarithm

Logarithm

For $a > 0$, $a \neq 1$, the function a^x is always strictly decreasing or increasing, and its range is the whole positive real line, for every positive real number b the equation

$$a^x = b$$

has a unique solution, denoted by $\log_a(b)$ (logarithm in base a of b). For every positive real number a we get a function

$$f(x) = \log_a(x),$$

- When the base is 10 we write simply $\log(x)$.
- When the base is e we write $\ln(x)$, this is the natural logarithm of x.

Properties of the logarithm

- For every real number x we have that $\log_a(a^x) = x$.
- For every positive real number x we have that $a^{\log_a(x)} = x$.
- For every a positive real number

$$\log_a(1) = 0,$$

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(x/y) = \log_a(x) - \log_a(y)$
- $\log_a(x^q) = q \log_a(x)$

Attention!!!

There is no "$\log_a$" in your calculator, but only $\ln$ or $\log$...

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} = \frac{\ln(x)}{\ln(a)}$$

Examples

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- Compute $\log_{2,01}(2)$ with your calculator (using the change of base formula).
- Simplify

$$\frac{\ln(2^{3/8})}{\ln(2^{-1/2})}$$

- Let $f(t) = 200(0{,}8)^t$. For which t we have $f(t) = 100$?

Active Learning: Solving the Core Lecture Problem

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Group Challenge: When Does an Investment Double? (4 mins)

Our initial goal (Slide 2): An investment yields 2% interest annually ($a = 1{,}02$). Capital over time follows: $K(t) = K_0(1{,}02)^t$. We want to find when the asset **doubles**: $K(t) = 2K_0 \implies 1{,}02^t = 2$.

Steps for Your Team:

1. Take the natural logarithm (ln) of both sides of the equation.
2. Apply the power log rule $\ln(x^q) = q\ln(x)$ to bring down time t.
3. Isolate t algebraically: $t = \frac{\ln(2)}{\ln(1{,}02)}$.

We will learn an economic shortcut (Rule of 70) when we go over Taylor polynomials

Making non-linear linear

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From Curve to Straight Line (Reviewing Slide 3)

The distribution of incomes follows a power function: $P(x) = Cx^a$ where $a < 0$. By applying logarithms to both sides, we yield a linear equation:

$$\log(P(x)) = a\log(x) + \log(C)$$

Desk Check:

Suppose you run a linear regression on national income data and get the following line:

$$Y = -1{,}5X + 12{,}4$$

where $Y = \log(P(x))$ and $X = \log(x)$.

- What is the exact value of the power law exponent parameter a?

Questions?

Thank you for your attention!

Polynomial Division

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Given a polynomial p, and a nonzero polynomial q it is always possible to find two polynomials k and r, with r either 0 or $\deg(r) < \deg(q)$ such that

$$p(x) = k(x)q(x) + r(x),$$

Application to finding roots

The real number a is a root of p if, and only if,

$$p(x) = k(x)(x - a) + 0,$$

Example

Find the roots of $p(x) = x^3 - 11x + 6$.

- Observe that 3 is a root.
- Divide $p(x)$ by $(x - 3)$.
- We get a polynomial of degree 2 for which we can always find the roots (if there are)

The Hunt for rational roots

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Useful fact

Let $p(x)$ be a polynomials with integer coefficients (that is $c_i \in \mathbb{Z}$). If rational number $\frac{m}{n}$ is a root for p, then we have that n divides the leading coefficient and m divides the constant term.

Example

Find all the rational roots of $p(x) = x^4 - 2x^3 - 14x^2 + 30x + 9$.

Summary

To find the roots of a polynomial of high degree:

- Look for easy roots.
- Divide.
- Repeat till you get a polynomial of degree 1 or 2.

Thank you for your attention!